Belief-Driven Secular Stagnation

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Abstract: This study constructs an endogenous growth model, which it then uses to examine the US economy, both before the 2008 recession and during the recovery period that followed. In particular, it explores the secular stagnation hypothesis and its implications for asset pricing. The model features technological externalities that imply multiple perfect-foresight balanced growth paths. In this setup, a change in agents' beliefs may trigger persistent slumps, low interest rates, and elevated risk premiums, consistent with the recent US experience. Numerically, using the Epstein and Zin preference, the model calibration suggests that the historical data moments can be accommodated by persistent regimes and a high intertemporal elasticity of substitution.

Key Words: Endogenous growth, Secular stagnation, Sunspots, Risk-free rate, Risk returns

JEL Codes: E22, E23, G10, G12

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1 Introduction

There is increasing evidence that global long-run growth has been declining since the recession of 2008. For example, Fernald & Jones (2014) document that economic growth in the United States is decelerating, along with growth in educational attainment, R&D intensity, and population. Antolin-Diaz et al. (2017) extend this narrative to other countries. This persistent reduction in long-run growth rates has led to a discussion of the secular stagnation hypothesis. For example, Gordon (2015) indicates that the economic engine on the supply side is gradually burning out. He argues that the pace of innovation is slowing down and that the labor force participation rate is decreasing permanently. Summers (2015) focuses on the demand side. He points out that the natural interest rate has declined, but that it cannot be implemented because of the zero lower bound (ZLB). This leads to a deviation of employment from the full employment level.

This study constructs an endogenous growth model in which secular stagnation may occur as a result of belief-driven self-fulfilling equilibria. The key setups in the model are as follows: (1) "AK" linear production; (2) the assumption that investment can feed back to productivity; and (3) extrinsic randomness. In theory, the model can generate an arbitrarily long period of stagnant growth, accompanied by a low investment level and weak productivity. In addition to the macro fundamental variables, the model has implications for movements of the risk-free rate and asset prices. Roughly speaking, the risk-free rate has dropped significantly in the United States during the past decade. Then, because there was no significant change in the risky asset return, the risk premiums increased accordingly. Section 2 describes the observations related to these variables. Cochrane (2005) also documents that risk premiums are large and volatile. However, the standard real business cycle (RBC) model has difficulties in explaining this. Among many others, Gourio (2012) and Gabaix (2012) consider a massive "disaster" shock in the RBC model, and successfully generate time-varying risk premiums. My results are similar to theirs in terms of the large downward shift in economic growth. However, the proposed model can also account for movements of the risk premiums and those of the risk-free rate. Thus, the stagnant period in the proposed modeled economy can be described as follows. A confidence shock hits the economy, causing investors and firms to become pessimistic about economic performance. As a result, firms cut their investments and, hence, the externalities of investment cannot maintain productivity at a healthy level. In return, this reinforces the firms' low investment strategy. The growth is trapped. Furthermore, the pessimistic outlook means consumers cannot expect high growth in consumption in the future. The asset return is low via the pricing mechanism. In summary, the model generates a dynamic system with (1) sustainable low growth and depressed investment, (2) a persistent decrease in productivity, (3) a decreasing trend in the risk-free rate, and (4) an increasing trend in risk premiums. The model is also calibrated to match historical data collected in the United States. With some adjustments to the baseline model, the calibration of the new model matches theoretical predictions based on 10 historical data moments calculated from the US data.

With regard to the setup of the model, studying secular stagnation requires a model that generates self-reinforcing slow growth. Here, the economic growth literature on "poverty traps" is helpful. For example, Azariadis & Stachurski (2005) examine numerous models with multiple equilibria. A widely used setup is a direct linkage between capital and productivity, where investment feeds back into productivity through channels such as complementarities, spillovers, and externalities. Azariadis & Drazen (1990) consider spillovers from human capital accumulation processes. Hence, they assume that productivity is a function of capital, which includes both human and physical capital. Here, I assume that technology is a threshold function of the investment-capital ratio. Technology jumps to a new level when the investment capital ratio reaches the "threshold." A negative demand shock on investment leads to fewer externalities and, therefore, weak productivity. On the other hand, firms set their investment according to productivity in order to maximize value. In a sense, the reduction in investment is path-dependent. This structure is a useful way to analyze secular stagnation because it fundamentally changes the trend of the economy. As in the literature, this assumption also offers multiplicity.

Lastly, to enable shifts between equilibria, I introduce regime-switching sunspots, which alter the beliefs and activate shifts among multiple balanced growth paths (BGPs) in the economy. The transition is governed by an exogenous Markov chain. This framework is carefully examined in Hamilton (1989). Similar setups are found in Benigno & Fornaro (2016) and Christiano & Harrison (1999). The former studies a Keynesian growth model with nominal rigidities, a ZLB on the interest rate, and confidence shocks. However, in contrast to their work, the proposed model considers real variables and incorporates risk premiums.

The remainder of the setup is standard in models of a production economy with complete markets. The model takes a general equilibrium approach. The firms are owned by households, but there are externalities in the production processes. I assume that firms face adjustment costs of investment. The optimal level of investment determines the BGP. On the consumer side, the model follows the standard consumption-based capital asset pricing model (CCAPM). As usual, under the complete market assumption and no arbitrage condition, the household's problem determines the unique stochastic discount factor (SDF).

This paper consists of two parts. The first part considers a baseline model with log-utility. The convenient mathematical properties of the model mean that propositions yielded under this preference are relatively elegant and easily understandable in terms of intuitive economic reasoning. These propositions construct the main theoretical analysis of the equilibrium conditions in this economy. However, the model with log-utility suffers a trade-off in its calibration, because the parameterizations that can generate proper macro fundamental moments cannot offer reasonable values of financial variables, and vice versa. Therefore, the second part of this paper attempts to improve the calibration by introducing the Epstein & Zin (1989) (EZ) preference. Although the EZ framework complicates the solutions and makes the corresponding propositions mathematically less elegant, most of the theoretical findings still hold. Moreover, the EZ preference is widely used to explain the behavior of the risk-free rate and the risk premium. By minimizing the loss function, the regime-switching model under the EZ preference can match all 10 historical data moments with reasonable parameter values. The calibration suggests that the model needs parameterizations that ensure highly persistent regimes and a high level of intertemporal elasticity of substitution (IES) to generate the collected data moments.

There are two main branches of literature related to this study. The first includes models used to study secular stagnation. In addition to Benigno & Fornaro (2016), there are many studies on the behaviors of interest rates with a ZLB, including Eggertsson et al. (2017), Pescatori & Turunen (2016), Gruber & Kamin (2016), Favero et al. (2016), and Sajedi & Thwaites (2016). These models focus mainly on the phenomenon that a low or negative natural rate of interest leads to a chronically binding ZLB. Essentially, they formalize the arguments of Summers (2015) in New Keynesian overlapping generation (OLG) models. They show that factors such as a slowdown in population growth, an increase in life expectancy, an increase in income inequality, and a decrease in the price of investment goods can reduce the natural interest rate. In contrast, I do not consider the ZLB, because I only consider real variables. More precisely, the proposed model examines trended long-run decreases in the real risk-free rate, but does not explain the bounded period of the nominal risk-free rate. Blanchard et al. (2017) argue that a low expectation of long-run productivity growth can affect output and inflation in the short run. They perform an empirical regression on data on forecasts of economic growth and forecast errors. From a theoretical perspective, their work employs similar logic to that employed here. Both consumers and firms, when pessimistic about future growth, tend to revise their behavior. Consumers modify their expectations of permanent income and firms change their investment plans. Hence, the downturn in the economy is self-reinforcing. However, their model mainly explores the mechanism in terms of unemployment, which differs from the focus of this work. Eichengreen (2017) compares the Great Recession of the 1930s and the more recent crisis in 2008. He compares the real GDP level of the year of the crisis with that eight years later, concluding that the recovery after the first crisis was faster than that of today. He argues that this is due to the fast adoption of new technologies, such as electricity and a national highway system, and the large demand for such technologies owing to the coming war. However, we do not see analogous processes today. Besides, there are studies attempt relate endogenous growth to asset prices. Among many others, Jinnai (2015) emphasis product cycle stemmed from transition from monopoly to perfect competition is a powerful amplification of risk premium.

Second, this study follows the model structure used in the literature, from two aspects. Many studies, such as Cochrane (1991), Jermann (1998), Boldrin et al. (2001), and Campbell (2003), have examined firms' investment behavior in a production economy. These studies focus on the relations between firms' investments, stock returns, and macroeconomic fluctuations. The general equilibrium structure in this study follows the literature in this area. Firms are owned by households. In equilibrium, the investor holds all stock and consumes the dividends. Given the pricing kernel, the firm decides on the investment for next period. Kogan & Papanikolaou (2012) survey the research in this field. In terms of methodology, closely related works include those of Kogan (2001) and Eberly & Wang (2009), who solve the central planner's problem in a continuous time framework. Other studies examine endogenous growth, such as the works of Fatas (2000) and Azariadis & Drazen (1990). The former considers the AK framework with cyclical shocks. This is, in some sense, a version of my model without the investment adjustment costs. However, it focuses on explaining persistent fluctuations in output. Azariadis & Drazen (1990) propose an assumption between technology and investment. Nonetheless, their work focuses mainly on "poverty traps" by linking human capital accumulation processes to productivity. With regard to sunspots, Benhabib & Farmer (1999) review the literature on various structures of the production function, nonlinear accumulation of capital, and extrinsic randomness in order to handle multiple equilibria.

This paper proceeds as follows. The next section describes the empirical observations addressed by this study. Section 3 presents the baseline model and its solution. In section 4, I introduce the assumption that endogenizes the technology. Section 5 constructs sunspot equilibria and presents several implications related to growth and asset pricing. Section 6 introduces the EZ framework, and updates the propositions in the baseline model. Section 7 recalibrates the model and offers several remarks. Section 8 concludes the paper.

2 Motivating Observations

This section presents a number of motivating observations, each of which is addressed by the proposed model.[1] Figures 2.1 and 2.2 display historical data on risk premiums, the risk-free rate, and per capita real GDP growth in the United States. In Figure 2.1, the dark bars are risk-free rates. The grey bars on top indicate the risk premium. The two parts add up to the risky asset return. Figure 2.2 shows the per capita real growth rate in the United States in recent decades. In addition, I construct the investment-to-GDP ratio using the difference between 1 and the aggregate consumption's share of GDP. Figure 2.3 plots the constructed ratio. The red shapes in the three figures roughly indicate the patterns that the model attempts to account for: stagnant growth, accompanied by weak investment; a dramatic fall in the risk-free rate; and an increase in risk premiums.

I divide the movements of these variables into two periods, either side of 2000. Before 2000, the average growth rate is approximately 2.5%. In addition, the risk

[1] Descriptions of the data are provided in appendix 9.12.

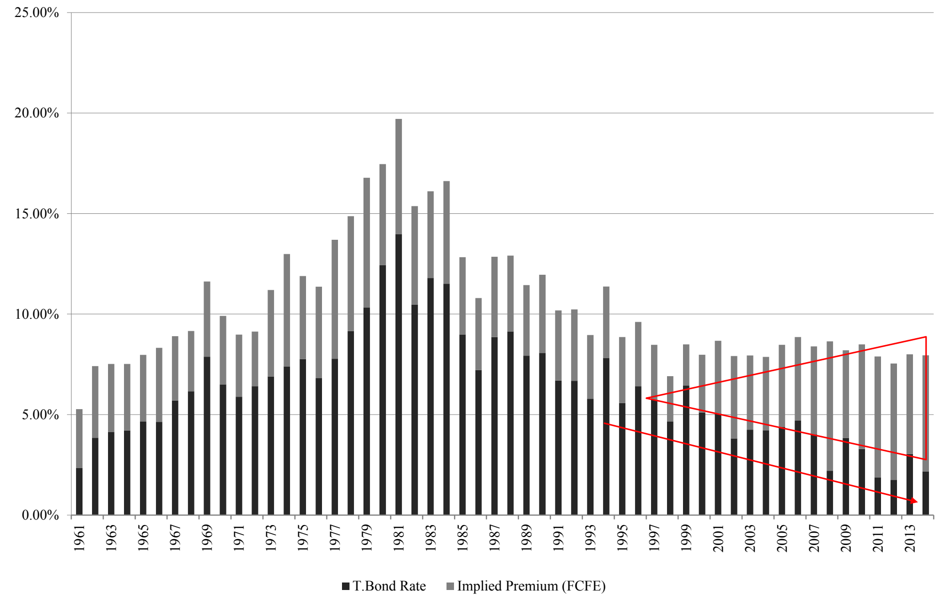

Figure 2.1: Historical risk premiums and risk-free rates in the United States

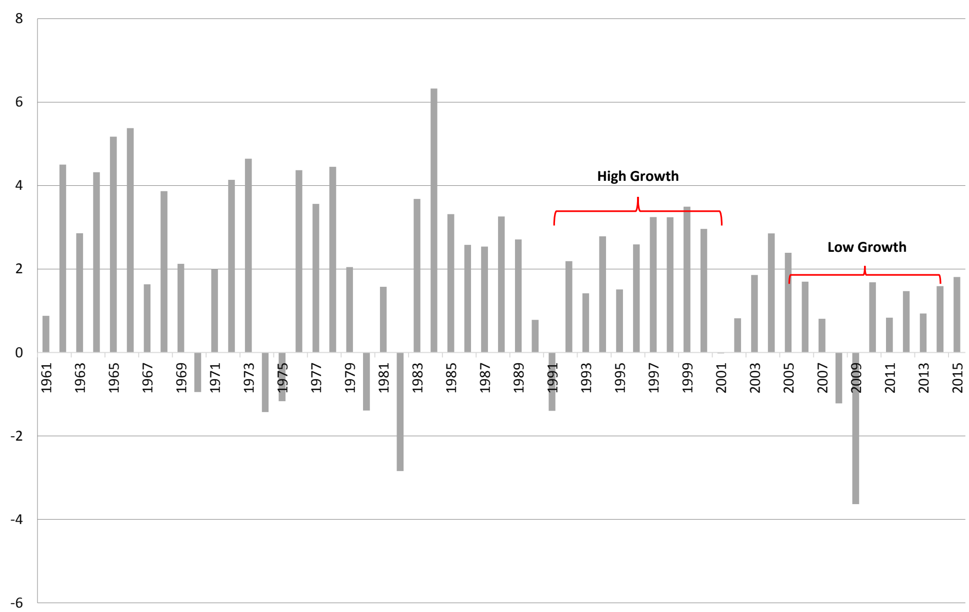

Figure 2.2: Historical per capita GDP growth in the United States

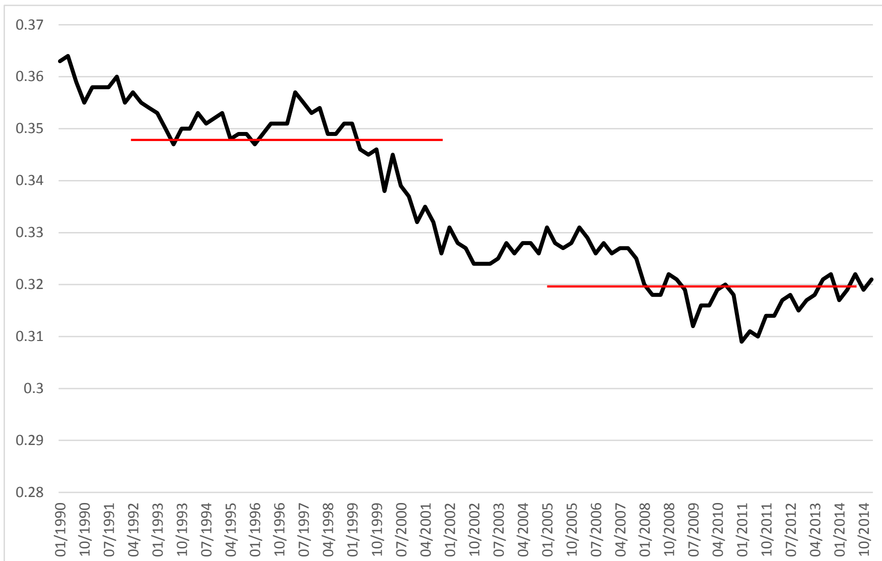

Figure 2.3: Constructed investment-to-GDP ratio in the United States

premium remains relatively stable. After 2000, the growth rate drops to around 1%, especially during the period after the crisis. Figure 2.3 shows a relatively clear decrease in the investment-to-GDP ratio. Visually, there is a structural break in the early 2000s. In the model, such breaks can be generated by a switch between two BGPs following a confidence shock (i.e. a sunspot shock), resulting in agents forming a pessimistic view about future growth. Accordingly, their best strategy is to reduce their investment, which is assumed to subsequently affect productivity. Then, the low productivity confirms that the low investment plan is optimal, and weak future growth is reinforced.

At the same time, the risk-free rate shows a noticeable drop, and risk premiums increase. These observations link directly to the well-known risk premium puzzle and risk-free rate puzzle, which can be traced back to Mehra & Prescott (1985). At a first-order approx around the steady-state, the conventional model predicts that the risk-free rate is $\tilde{r}_{ft} \approx RiskAversion \times E_t\left[\Delta\tilde{c}_{t+1}\right]$. This predicts volatile risk-free rate, which is counterfactual, especially if risk aversion is large. In the model, this could also be the result of pessimism. Then, because consumers cannot expect high growth in future consumption, the pricing mechanism indicates that the risk-free rate will be low. In addition, the adjustment cost of investment and the structure of the stochastic regime-switching framework also help to explain the increase in risk premiums.

Specifically, I use the period 1992 to 2001 as a high-growth period, with a relatively high risk-free rate and a low risk premium. In contrast, the period 2005 to 2014 is employed as a low-growth period, with a low risk-free rate and a large risk premium. Later, calibrations are carried out based on the data collected from these two periods.

3 Baseline Model and Solution

This section solves the baseline model to establish the relation between a firm's investment and technology. The model describes an economy with one productive sector. Time t runs discretely from zero to infinity. There are many identical firms and consumers. I consider a state variable s_t. Let $s^t = (s_0, s_1, \ldots, s_t)$ be the notation of the history of the state variable. Here, $\{s_t\}_0^\infty$ is a stochastic process. The probability distribution of $\{s_t\}_0^\infty$ is left for later discussion. All endogenous variables introduced later are functions of the histories s^t. The production function is linear, given as $Y(s^t) = A(s^t)K(s^t)$, where Y, K, and A denote the output, capital stock, and technology scale factor, respectively. A firm uses its operating profit to pay dividends, as $D(s^t) \equiv A(s^t)K(s^t) - I(s^t)$, where I is the investment. Additionally, I restrict dividends to be positive; that is, $D(s^t) > 0$. The representative firm maximizes its stock value, represented by the discounted cash flows

$$V\left(s^t\right) = \underset{I}{Max} \quad E_t\left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} \frac{\Lambda\left(s^\tau\right)}{\Lambda\left(s^t\right)} D\left(s^\tau\right)\right], \tag{3.1}$$

subject to the constraints

$$\frac{K\left(s^{t+1}\right)}{K\left(s^{t}\right)} = g\left(i\left(s^{t}\right)\right) \tag{3.2}$$

$$A\left(s^{t}\right) > 0, \quad K\left(s^{t}\right) > 0, \quad \Lambda\left(s^{t}\right) > 0 \tag{3.3}$$

$$V\left(s^{t}\right) > 0, \quad I\left(s^{t}\right) > 0 \tag{3.4}$$

$$K\left(s^{0}\right) \textit{ is given}, \tag{3.5}$$

where E_t is the mathematical expectation based on information in time t, and $\beta \in (0,\ 1)$ is the time preference parameter. For simplicity, I denote the investment capital ratio as $i \equiv I/K$. In addition, β and Λ together constitute the discount factor. The firm is a price taker, and takes the discount factor as given. The equation (3.2) is the capital accumulation condition. The function $g(\cdot)$, which I refer to as the efficiency function of an investment, captures the effectiveness of converting an investment into capital inputs. To understand this function, consider the extreme case of $g(i) = i$. The capital accumulation condition becomes $K_{t+1} = I_t$. Here, the investment has no adjustment costs and is completely efficient. However, in the literature, the standard assumption is that the adjustment costs of an investment are convex. That is, the more we invest, the more the investment costs. Therefore, I restrain the function using $g(i) > 0$, $1 > g'(i) > 0$, and $g''(i) \leq 0$.[2] Furthermore, I assume that the efficiency function is homogeneous of degree one in I and K, following the proposition of Hayashi (1982). This proposition simplifies the model and helps to derive the findings related to asset prices.

There is a broad range of literature on linear production, constant returns to scale, and externalities. Here, I borrow the intuitive reasoning described in Azariadis & Drazen (1990). First, there is a distinction between private and public factors of production, as introduced by Romer (1986). Private factors are controlled by individual firms. Public factors are not controlled by any specific producer. In the production process, there are spillovers from the private capital factor to the public capital factor. On an aggregate level, the productivity scale A consists of both factors. Because of these externalities, production in the economy shows constant returns to scale.

On the other hand, the households own the firms and face the consumer's prob-

[2]See Appendix 9.1 for further details.

lem. Specifically, the total shares of stock are normalized to unity. In this case, the representative consumer faces a standard infinite-horizon utility-maximization problem, given by

$$J\left(s^{t}\right)=\underset{C}{Max} \quad E_{t}\left[\sum_{\tau=t}^{\infty} \beta^{\tau-t} U\left(C\left(s^{\tau}\right)\right)\right], \tag{3.6}$$

subject to the budget constraint

$$S\left(s^{t+1}\right) P\left(s^{t}\right)=S\left(s^{t}\right)\left[P\left(s^{t}\right)+D\left(s^{t}\right)\right]-C\left(s^{t}\right), \tag{3.7}$$

where S denotes the shares of stock held by consumers, P is the asset price, and C denotes consumption.

The model is closed by the resource constraint,

$$C\left(s^{t}\right)=D\left(s^{t}\right). \tag{3.8}$$

For consumers, the model is a standard consumption-based capital asset pricing model (CCAPM), which can be traced back to Lucas (1978). In equilibrium, the investor holds the single asset and consumes its dividends. The asset price P maps to the stock value V in the firms' problem. However, the problems are slightly different. To follow the frequently used notation, P denotes the ex-dividend price, which fulfills $V_t = P_t + D_t$.

In next subsection, I first solve for the first-order conditions on both sides. The general equilibrium is provided by the combination of the first-order conditions and the market clearing condition.

3.1 General Equilibrium Condition

The standard method of dynamic programming can be used to derive the optimal conditions. Proposition 1 provides the key results for the firms' problem. For simplicity, I use $X\left(s^{t}\right)$, X_t, and the simplified notation X interchangeably when there is no ambiguity.

Proposition 1. *The firms' problem has the following first-order conditions (FOCs) and Euler equation.*

The first-order conditions with respect to investment I and capital K are

$$E_t\left[\beta\frac{\Lambda_{t+1}}{\Lambda_t}\frac{\partial V_{t+1}}{\partial K_{t+1}}\right]=\frac{1}{g'\left(i_t\right)} \tag{3.9}$$

$$\frac{V_t}{K_t}=A_t-i_t+\frac{g\left(i_t\right)}{g'\left(i_t\right)}. \tag{3.10}$$

The Euler equation is

$$E_t\left[\beta\frac{\Lambda_{t+1}}{\Lambda_t}\left(A_{t+1}-i_{t+1}+\frac{g\left(i_{t+1}\right)}{g'\left(i_{t+1}\right)}\right)g'\left(i_t\right)\right]=1. \tag{3.11}$$

Proof. See Appendix 9.2.

The prime symbol $'$ denotes a derivative, such as $g'(i)\equiv\partial g(i)/\partial i$ and $g''(i)\equiv\partial^2 g(i)/\partial i^2$. I follow convention and refer to the marginal price of capital on the left-hand side of FOC (3.9) as "marginal q." I also use "average Q" to indicate the average price of capital, V/K.

The FOC with respect to investment I shows that the firm's value is maximized when the investment balances the marginal gain and the marginal loss to the firm value. The proposition derived by Hayashi (1982) helps the model to obtain the FOC (3.10) of capital K. For a problem such as this, the Hayashi (1982) proposition confirms that the "marginal q" ($\partial V/\partial K$) is equal to the "average Q" (V/K).[3] Equation (3.10) shows that, given the predetermined capital stock K_t, the firm's value V is not monotonically related to the investment I. First, $A-i$ represents the dividends in the current period. High investment tends to decrease the current dividend payment. Therefore, the asset is less attractive. Next, the investment I enters the second term. This term is constructed as the growth rate of capital $g(i)$ over $g'(i)$, which measures the marginal effectiveness of the investment. Clearly, the second term is positively related to the investment-capital ratio i. As a result, there is a trade-off between the two terms. A firm with higher i has higher growth, but does not necessarily have higher Q. However, the implication for the asset return is not clear until the model is solved for the general equilibrium.

The Euler equation (3.11) is a stochastic differential equation that defines the path of the firm's optimal investment behavior, given the process of discount factor Λ and the technology scale A.

[3] See Appendix 9.3 for the proof.

On the other hand, the consumer's problem is standard, and provides the well-known Euler equation, given by

$$P_t U'(C_t) = E_t\left[\beta U'(C_{t+1})(P_{t+1} + D_{t+1})\right]. \tag{3.12}$$

The Euler equation of the consumer side offers the discount factor $\Lambda = U'(C)$.[4] Hence, I offer the following definition of a general equilibrium for the economy.

Definition 1. An equilibrium is a set of sequences $K^*(s^t)$, $I^*(s^t)$, $i^*(s^t)$, $D^*(s^t)$, $C^*(s^t)$, $S^*(s^t)$, $A(s^t)$, $\Lambda(s^t)$, and $V(s^t)$, such that:

1. $C^*(s^t)$ and $S^*(s^t)$ solve the household's optimization problem (3.6), given $V(s^t)$ and $D^*(s^t)$.
2. $K^*(s^t)$, $I^*(s^t)$, $i^*(s^t)$, and $D^*(s^t)$ solve the firm's problem (3.1), given $A(s^t)$, $\Lambda(s^t)$, and initial capital stock K_0.
3. $\Lambda(s^t)$ is the unique discount factor that satisfies $\Lambda(s^t) = U'(C^*(s^t))$.
4. The markets clear: $C^*(s^t) = D^*(s^t)$.
5. The transversality condition holds.[5]

As usual, I combine the optimal conditions on both sides to obtain the general equilibrium condition. For simplicity, this study considers the log-utility and, therefore, sets $\Lambda = C^{-1}$. Using the market clearing condition $C = D = AK - I$, I rearrange equation (3.11) as follows:

$$E_t\left[\beta\left(\frac{(A_{t+1} - i_{t+1})K_{t+1}}{(A_t - i_t)K_t}\right)^{-1}\left(A_{t+1} - i_{t+1} + \frac{g(i_{t+1})}{g'(i_{t+1})}\right)g'(i_t)\right] = 1. \tag{3.13}$$

This is the core stochastic difference equation governing the dynamics of a firm's optimal decisions. Using the initial values, the firm recursively solves the equation to follow the optimal path. By solving this equation, I obtain several findings on macroeconomic fundamentals and asset prices in this economy.

[4]See Appendix 9.4 for the derivation.
[5]See Appendix 9.5 for details.

3.2 Perfect Foresight Equilibrium

This section offers three propositions to illustrate the perfect foresight equilibrium. Proposition 2 shows that there are no transitional dynamics in the equilibria of the model. Based on this, Proposition 3 reveals the interactions between the exogenous technology scale A and the investment-to-capital ratio i. Then, Proposition 4 develops solutions for the risk premium, expected risk asset return, and risk-free rate. First, I study the deterministic model, which ignores the stochastic parts and the expectation operator in the model. Accordingly, the difference equation (3.13) is given by

$$\beta \left[\left(\frac{A - i_{t+1}}{A - i_t} \right) g(i_t) \right]^{-1} \left(A - i_{t+1} + \frac{g(i_{t+1})}{g'(i_{t+1})} \right) g'(i_t) = 1. \tag{3.14}$$

The following propositions are constructed based on this difference equation. First, Lemma 1 offers the transversality condition in this deterministic model. This helps to develop the first proposition, which states that the deterministic equilibrium has no transitional dynamics.

Lemma 1. *The transversality condition for the deterministic version of the baseline model is given by*

$$\lim_{t \to \infty} \left[\beta^t \frac{g(i_t)}{(A - i_t)\, g'(i_t)} \right] = 0. \tag{3.15}$$

Proof. See Appendix 9.5.

Proposition 2. *With the condition*

$$\frac{g(0)}{A g'(0)} < \frac{\beta}{1 - \beta}, \tag{3.16}$$

the model has the following proposition.

In all paths governed by the difference equation (3.14), the only feasible path for the model is that in the fixed point where the investment-to-capital ratio $\bar{i}$ solves

$$\frac{g(\bar{i})}{g'(\bar{i})(A - \bar{i})} = \frac{\beta}{1 - \beta}. \tag{3.17}$$

That is, there are no transitional dynamics toward an equilibrium.

Proof. See Appendix 9.6.

Because the model has the only feasible path in the fixed point, in some sense, the model is static. In the deterministic difference equation (3.14), technology A is a constant. Nonetheless, it is straightforward to generalize the key equation in Proposition 2 to the case allowing A to be a stochastic process:

$$\frac{g\left(\bar{i}_t\right)}{g'\left(\bar{i}_t\right)\left(A_t - \bar{i}_t\right)} = \frac{\beta}{1-\beta}. \tag{3.18}$$

Overall, when the firm observes the current realization of the technology scale A, it solves equation (3.18) and sets its investment-to-capital ratio i to $\bar{i}_t$.[6] In this situation, i follows the dynamics of the technology process A. Moreover, in this BGP, consumption grows at the rate of capital growth, because $C_{t+1}/C_t = \left(A_{t+1} - i_{t+1}\right) K_{t+1} / \left(A_t - i_t\right) K_t$. Nevertheless, the functional form of the nonlinear efficiency function $g(\cdot)$ is not well established. Hence, I cannot solve the equation (3.18) explicitly. However, Proposition 3 illustrates the relation between the investment-to-capital ratio i and technology A in the implicit function (3.18). In addition, for later reference, I use $i[A]$ to denote the solution of a given A.

Proposition 3. *Given the same conditions as those in Proposition 2, the implicit function between A and i (i.e., equation (3.18)) is characterized by the features $1 > \partial i / \partial A > 0$ and $\partial^2 i / \partial A^2 < 0$.*

Proof. See the middle part of Appendix 9.7.

Proposition 3 shows that if the technology scale A increases, the investment-to-capital ratio i increases. The firm chooses a higher investment to cope with a higher productivity parameter A. Moreover, this boosts the growth rate of the economy because the growth is determined by $g(i)$. Nonetheless, $\partial^2 i / \partial A^2 < 0$ tells that the impact of technology A on economy growth becomes less powerful in the model over time, owing to the concavity of the efficiency function $g(i)$. Intuitively, with an increase in technology A, the firm wants to increase its investment to obtain the optimal value. However, the adjustment costs are high when the investment level is high. Roughly speaking, a significant investment is "wasted" and cannot be transferred into capital inputs.

[6] See the first part of Appendix 9.7 for a discussion of the root.

Moreover, with the solution $i[A]$, the model yields expressions for the finance-related variables in Proposition 4.

Proposition 4. *When the technology scale A follows a stochastic process, conditioned on the current state s^t, the stochastic version of the baseline model has the following solutions for the expected risk premium $E_t(RP_{t+1})$, expected risky asset return $E_t(R_{t+1})$, and the risk-free rate r_t^f:*

$$E_t\left(R_{t+1}|\, s^t\right) = \frac{g'(i[A_t])}{1-\beta} E_t\left[A_{t+1} - i[A_{t+1}]|\, s^t\right] \tag{3.19}$$

$$r_t^f = \frac{g'(i[A_t])}{1-\beta}\left[E_t\left(\frac{1}{A(s^{t+1}) - i[A(s^{t+1})]}\middle|\, s^t\right)\right]^{-1} \tag{3.20}$$

$$E_t\left(RP_{t+1}|\, s^t\right) = \frac{g'(i[A_t])}{1-\beta}\left\{E_t\left[A_{t+1} - i[A_{t+1}]|\, s^t\right] - \left[E_t\left(\frac{1}{A(s^{t+1}) - i[A(s^{t+1})]}\middle|\, s^t\right)\right]^{-1}\right\}. \tag{3.21}$$

Proof. See Appendix 9.9.

Proposition 4 offers solutions for asset returns, the risk-free rate, and risk premiums. All three expressions are functions of the stochastic process A_t when $i[A]$ is replaced by the root of equation (3.18). In the deterministic case, one can ignore the expectation operator. As a result, the risky asset return and risk-free rate collapse to one expression.

Importantly, the expression for the risk premium reveals that it can move counter-cyclically in the modeled economy. The risk premium consists of two parts, namely, $g'(i[A_t])/(1-\beta)$ and the term in the bracket. For now, I assume that s^t is independent and identically distributed (i.i.d.) in order to exclude the impact of the second term and to consider the effect of the first part only. Later, the second part will be added. Given this assumption, the term in the brace of function (3.21) is a constant. Owing to the concavity assumption $g''(i) < 0$, in the economy with a higher investment-to-capital ratio i and higher growth, the term $g'(i[A_t])/(1-\beta)$ is actually smaller. Therefore, we observe a higher risk premium in the economy with slow growth in this situation. Here, the counter-cyclical risk premium is caused by the adjustment costs of investment. The extreme case demonstrates this. When $g(i) \to i$, there is no cost for investment. Accordingly, $g'(i) \to 1$. Here, the risk premium becomes

$$RP\left(s^t\right) = \frac{1}{1-\beta}\left\{E_t\left[A_{t+1} - i\left[A_{t+1}\right]\middle| s^t\right] - \left[E\left(\frac{1}{A\left(s^{t+1}\right) - i\left[A\left(s^{t+1}\right)\right]}\middle| s^t\right)\right]^{-1}\right\}, \tag{3.22}$$

which is a constant under the i.i.d. assumption. The counter-cyclicality of the risk premium disappears.

In addition, the expectation terms in the solutions are of interest. In later sections, I release the i.i.d. assumption and study the second parts in the solutions. If the distribution of A_{t+1} relates to A_t, the movement of the risk premium is affected by the expectations in the bracket. This could be a second source of the counter-cyclical behavior.

4 Endogenous Productivity and Multiple Equilibria

The baseline model shows the conventional logic of economic growth. The firms decide on their investments based on their productivity. Furthermore, the level of investment determines the growth rate. In this section, I endogenize technology. This methodology originates in the growth literature on poverty traps. Azariadis & Stachurski (2005) provide a good survey on this field. Often, the technology scale A is no longer exogenous in such models, and is determined by a function of capital K or of investment I. As such, the convex neoclassical growth model generates multiple BGPs. To some extent, this explains the self-reinforced poverty evident in many developing countries. In addition, many studies attempt to provide a micro foundation to the relation between technology and investment. Among many others, Matsuyama (1997) summarizes a series of papers that discuss the feedback of investment to externalities and productivity, owing to imperfect competition and complementarity. However, the structures of these models complicate the parsimonious model proposed here. As in Azariadis & Drazen (1990), I skip the micro foundation and assume a simple relation between technology and investment in assumption 1. Explicitly, there is a discontinuous relation between the productivity scale factor A and the investment-to-capital ratio i.

Assumption 1. *The technology scale factor A is a discontinuous function of the* *investment capital ratio i, and is given by*

$$A(i_t) = \begin{cases} A_H; \; if \; i_t \geq i^*, \; or \; equivalently \; s^t \geq s^* \\ \\ A_L; \; if \; i_t < i^*, \; or \; equivalently \; s^t < s^* \end{cases} \tag{4.1}$$

I define i_H and i_L as the solutions to the equation (3.17) *with A_H and A_L, respectively; that is, $i_H = i[A_H]$ and $i_L = i[A_L]$.*

This assumption describes a threshold relation. The investment-to-capital ratio should be maintained above a certain level, i^*, to ensure a high performance of technology level A_H. Otherwise, the technology is trapped in a relatively low level A_L. Clearly, this assumption ensures there are two solutions to equation (3.17), namely $i[A_H]$ and $i[A_L]$, given A_H and A_L, respectively. There are two equilibria: $\{A_H, \; i_H\}$ and $\{A_L, \; i_L\}$, for the firms to select.

Intuitively, the more a firm invests, the more spillovers there are to the public factor of production. Hence, the economy exhibits higher productivity. In turn, a high technology scale A urges firms to investment more. For instance, if all firms choose the high investment-to-capital ratio i_H, this generates a high technology scale A_H. In turn, A_H confirms that i_H is the optimal choice for an individual firm. Because we endogenize the technology, it is of little interest to identify which of the two determines the other. The assumed co-movement is clear. The technology A stands for the supply side, and the investment stands for the demand side. They are interconnected.

With Assumption 1, the baseline model exhibits multiple BGPs. More importantly, these two BGPs correspond to different growth rates, risk-free rates, and risk premiums. Obviously, a BGP with a low investment-to-capital ratio i_L grows at a slower pace. The economy is trapped permanently if there is no "shifting device" in the economy.

Furthermore, I combine the findings in Proposition 4 with those of Assumption

1, yielding

$$E_t\left(R_{t+1}\middle| i_t = i_L\right) = \frac{g'\left(i_L\right)}{1-\beta} E_t\left[A_{t+1} - i\left[A_{t+1}\right]\middle| i_t = i_L\right] \tag{4.2}$$

$$E_t\left(R_{t+1}\middle| i_t = i_H\right) = \frac{g'\left(i_H\right)}{1-\beta} E_t\left[A_{t+1} - i\left[A_{t+1}\right]\middle| i_t = i_H\right] \tag{4.3}$$

$$\left[r_t^f\middle| i_t = i_L\right] = \frac{g'\left(i_L\right)}{1-\beta}\left[E\left(\frac{1}{A\left(s^{t+1}\right) - i\left[A\left(s^{t+1}\right)\right]}\middle| i_t = i_L\right)\right]^{-1} \tag{4.4}$$

$$\left[r_t^f\middle| i_t = i_H\right] = \frac{g'\left(i_H\right)}{1-\beta}\left[E\left(\frac{1}{A\left(s^{t+1}\right) - i\left[A\left(s^{t+1}\right)\right]}\middle| i_t = i_H\right)\right]^{-1}. \tag{4.5}$$

As predicted by Proposition 4, with the i.i.d assumption, we have

$$E_t\left(R_{t+1}\middle| i_t = i_L\right) > E_t\left(R_{t+1}\middle| i_t = i_H\right) \tag{4.6}$$

$$\left[r^f\middle| i_t = i_L\right] > \left[r^f\middle| i_t = i_H\right]. \tag{4.7}$$

When the technology scale A is high, firms invest more in order to optimize their stock value. High investment promises good future growth, which makes the asset more attractive. Nonetheless, the high adjustment costs harm consumption growth and the asset prices. In this simple setup, the second force dominates the first. Hence, the model yields a low asset return in the high growth state.

This intuitive reasoning is clearly not consistent with the observations shown in Figure 2.1. In the years after 2008, we see relatively low growth, accompanied by a drop in the risk-free rate, which does not agree with the results of equation (4.7). However, note that the effects of these expectation terms have not been explored. In next section, I relax the i.i.d. assumption. In the end, the risk premium in this model is driven by the interaction between the efficiency function $g(i)$ and the expectation of the difference between the technology A and the investment-to-capital ratio i.

Henceforth, I consider s_t as a nonfundamental extrinsic shock. Following conventions, I use the term "sunspots." Intuitively, s^t serves as a selection device that begins the endogenous chain reactions. The next subsection introduces the sunspots in a formal way.

5 Sunspot Equilibria

This section defines the sunspots and studies their implications in the updated model. Among many others, Woodford (1986) explores a continuum of sunspot equilibria that asymptotically converge to the BGP. However, the sunspots defined below are in different environments to those of the mainstream literature. Here, the sunspot equilibria are not near the indeterminate equilibrium, and are similar to those used in Benigno & Fornaro (2016) and Christiano & Harrison (1999). In this framework, a sunspot is a signal in a period that guides a firm to choose an equilibrium for the current period. As indicated by Christiano & Harrison (1999), a proper name for this kind of extrinsic randomness is regime-switching sunspots. Specifically, I provide assumption 2.

Assumption 2. *s_t is an extrinsic random variable that governs the system and acts as a selection device in the model. $\{s_t\}_0^\infty$ is a Markov chain with state space $\{s_L,\ s_H\}$ and transitional matrix $\mathbf{P}_s$, given by*

$$\begin{pmatrix} p_L & 1-p_L \\ 1-p_H & p_H \end{pmatrix}. \tag{5.1}$$

For example, if $s_t = s_L$, the probability of shifting to the other state s_H is

$$Prob\,(\,s_{t+1} = s_H |\, s_t = s_L) = 1 - p_L. \tag{5.2}$$

Explicitly, given s_t, we draw s_{t+1}. The realization of s_{t+1} determines the equilibrium in which the economy stays, $\{A_H,\ i_H\}$ or $\{A_L,\ i_L\}$. The two equilibria represent BGPs with a high growth rate and a low growth rate, respectively. Similarly, Gourio (2012) discusses two states and state-shifting, although he assigns two states with different technology shock processes. Here, the two states are defined by distinct equilibria.

In fact, s_t can represents a symbol of beliefs. At the beginning of each period, firms observe a signal s_L or s_H. Accordingly, they choose $\{A_H,\ i_H\}$ or $\{A_L,\ i_L\}$. The economy grows at rate $g\,(i_H)$ or $g\,(i_L)$, respectively. The appearance of s_L or s_H is exogenous and governed by a Markovian property. Within this setup, the economy switches between two BGPs. This parsimonious structure generates many patterns in growth and in asset pricing. For example, if p_L and p_H have relatively high values, the economy has a high probability of lingering in its current state.

In last section, the model's predictions under the i.i.d. assumption are set against the observations. Here, the i.i.d. assumption is replaced by Assumption 2. As such, the solutions of the expected risk return and the risk-free rate in this framework are given by

$$E\left[R_{t+1}\middle| s_t = s_L\right] = \frac{g'(i_L)}{1-\beta}\left[AiH - p_L\left(AiH - AiL\right)\right] \tag{5.3}$$

$$E\left[R_{t+1}\middle| s_t = s_H\right] = \frac{g'(i_H)}{1-\beta}\left[AiL + p_H\left(AiH - AiL\right)\right] \tag{5.4}$$

$$\left[r_t^f\middle| s_t = s_L\right] = \frac{g'(i_L)}{1-\beta}\left[\frac{1}{AiH} - p_L\left(\frac{1}{AiH} - \frac{1}{AiL}\right)\right]^{-1} \tag{5.5}$$

$$\left[r_t^f\middle| s_t = s_H\right] = \frac{g'(i_H)}{1-\beta}\left[\frac{1}{AiL} + p_H\left(\frac{1}{AiH} - \frac{1}{AiL}\right)\right]^{-1}, \tag{5.6}$$

where $AiL \equiv A_L - i_L$ and $AiH \equiv A_H - i_H$.[7]

If p_L is equal to $1 - p_H$, the result is the same as the i.i.d. sunspot case. Using different specifications of AiH, AiL, p_H, and p_L, these Markovian sunspots generate different kinds of risk premium patterns. I present Proposition 5 to illustrate the possible outcomes.

Proposition 5. *Given Assumptions 1 and 2, if p_L, p_H, and/or $AiH - AiL$ are sufficiently large, the model can obtain the following:*

(1) a pro-cyclical expected risk return, such that $E\left[R_{t+1}\middle| s_H\right] > E\left[R_{t+1}\middle| s_L\right]$,

(2) a pro-cyclical risk-free rate, such that $\left[r_t^f\middle| s_H\right] > \left[r_t^f\middle| s_L\right]$,

simultaneously.

Proof. Before the main body of the proof, I state two points. First, the condition $AiH > AiL > 0$ is proved in the last part of Appendix 9.7. Hence, $(1/AiH) - (1/AiL)$ is negative. This condition is used repeatedly. Second, the phrase "second term" used in this proof refers to the term in equations (5.3), (5.4), (5.5), and (5.6), excluding $g'(i)/(1-\beta)$. For example, the second term in equation (5.3) is $[AiH - p_L(AiH - AiL)]$.

I start the proof by holding $AiH - AiL$ constant and discussing the probability parameters, p_L and p_H. When p_L increases, given that $(1/AiH) - (1/AiL)$ is negative,

[7] See Appendix 9.8 for the derivation.

the second terms in equations (5.3) and (5.5) decrease. For p_H, the opposite is true in equations (5.4) and (5.6).

Admittedly, in equation (5.3) and equation (5.4), we have $g'(i_L) > g'(i_H)$. Despite this, if p_L and p_H are sufficiently high, it could that the second terms dominate, yielding $E\left[R_{t+1}|\, s_H\right] > E\left[R_{t+1}|\, s_L\right]$.

In equations (5.5) and (5.6), because $(1/AiH) - (1/AiL)$ is negative, it is also true that if p_L and p_H are sufficiently high, we have

$$\left[\frac{1}{AiH} - p_L\left(\frac{1}{AiH} - \frac{1}{AiL}\right)\right]^{-1} < \left[\frac{1}{AiL} + p_H\left(\frac{1}{AiH} - \frac{1}{AiL}\right)\right]^{-1}. \tag{5.7}$$

Again, if this effect dominates the effect of $g'(i_L) > g'(i_H)$, the model results in $\left[r_t^f\middle|\, s_H\right] > \left[r_t^f\middle|\, s_L\right]$.

Next, $AiH - AiL$ offers a similar conclusion if the probability parameters p_L and p_H are held constant. For equations (5.3) and (5.4), we employ identical logic to that of the last argument. For equations (5.5) and (5.6), a larger difference between AiH and AiL means a larger difference between the reciprocals of AiH and AiL, owing to the monotonic feature of the reciprocal function. Furthermore, $(1/AiH) - (1/AiL)$ is negative. Hence, a larger $AiH - AiL$ means a smaller $(1/AiH) - (1/AiL)$. Again, a sufficiently large $AiH - AiL$ yields the following result:

$$\left[\frac{1}{AiH} - p_L\left(\frac{1}{AiH} - \frac{1}{AiL}\right)\right]^{-1} < \left[\frac{1}{AiL} + p_H\left(\frac{1}{AiH} - \frac{1}{AiL}\right)\right]^{-1}. \tag{5.8}$$

Furthermore, if this effect dominates that of $g'(i_L) > g'(i_H)$, the model yields $\left[r_t^f\middle|\, s_H\right] > \left[r_t^f\middle|\, s_L\right]$.

Q.E.D. ■

In addition to these possibilities, the extreme case also helps to illustrate the following argument. When $p_L \to 1$, $A_L \to 0$, and $i_L \to 0$, we have $\left[r_t^f\middle|\, s_t = s_L\right] \to 0$, and economic growth approaches the lowest level $g(0)$. In this case, there is no incentive for firms to invest. Therefore, the technology does not have enough development to pull the economy out of the low-state BGP. Investors have low expectations in related to consumption growth, and the risk-free rate approaches the ZLB.

For the counter-cyclicality of the risk premium, the model is also indeterminate, because it depends on the interactions between the marginal efficiency of the investment $g'(i)$ and the probabilities p_L, p_H, and $AiH - AiL$. In other words, the model should have sufficient degrees of freedom to generate a (1) pro-cyclical expected risk return $E\left[R_{t+1}\middle| s_H\right] > E\left[R_{t+1}\middle| s_L\right]$, (2) a pro-cyclical risk-free rate $\left[r_t^f\middle| s_H\right] > \left[r_t^f\middle| s_L\right]$ and (3) a counter-cyclical risk premium

$$E\left[R_{t+1}\middle| s_H\right] - \left[r_t^f\middle| s_H\right] < E\left[R_{t+1}\middle| s_L\right] - \left[r_t^f\middle| s_L\right], \tag{5.9}$$

simultaneously.

To summarize, first, expectations are crucial to the BGP level investment, technology, and long-run growth in the model. Moreover, when pessimism dominates, the economy tends to be trapped in a stagnated state with low technology, low investment, low growth, and a risk-free rate bounded by zero.

Nevertheless, Proposition 5 shows the potential of generating a pattern in the data with basic log preferences. Appendix 9.10 attempts to pre-form preliminary numeric exercises using the basic model. Though a log preference is able to capture some characteristics in the data, it also leaves some parts unexplained. In the next section, I update the model to the EZ framework and calibrate the model.

6 The Model in the EZ Framework

This section introduces the EZ utility into the baseline model. The firm's problem and its Euler equation remain the same as those discussed in section 3. For the consumers' problem in the EZ framework, the linear time-separable preference is generalized into a nonlinear function given by

$$J_t = \left\{ C_t^{1-\rho} + \beta \left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1-\rho}{1-\gamma}} \right\}^{\frac{1}{1-\rho}}, \tag{6.1}$$

where ρ is the reciprocal of the intertemporal elasticity of substitution (IES) of deterministic variations, and γ is the risk aversion coefficient. If $\rho = \gamma$, it reduces to the power utility. Basically, the EZ framework separates ρ from γ. The discount factor in this case is well established, as shown in Weil (1989) and Cochrane (2005).

The derivation is provided in Appendix 9.13.

$$DF_{t+1} = \left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right]^{\theta}\left(\frac{1}{R_{t+1}}\right)^{1-\theta}, \tag{6.2}$$

where R represents the return on the wealth portfolio and $\theta \equiv (1-\gamma)/(1-\rho)$. Similar to those general equilibrium models in a production economy, the discount factor plays a vital role in the optimal condition of the equilibrium.

Proposition 6. *In the updated model with the EZ preference, I join the first-order condition from the firm and the consumer to obtain the following optimal stochastic difference equation:*

$$E_t\left\{\left[\beta\left(\frac{A_{t+1}-i_{t+1}}{A_t-i_t}\cdot g(i_t)\right)^{-\rho}\left(A_{t+1}-i_{t+1}+\frac{g(i_{t+1})}{g'(i_{t+1})}\right)g'(i_t)\right]^{\theta}\right\}=1 \tag{6.3}$$

Second, the expressions for stochastic discount factor SDF_{t+1}, risk free rate r_t^f, and the expected risky asset return $E_t(R_{t+1})$, are given by

$$SDF_{t+1} = \beta^{\theta}\left(\frac{A_t-i_t}{g(i_t)(A_{t+1}-i_{t+1})}\right)^{\theta\rho}\left[\left(A_{t+1}-i_{t+1}+\frac{g(i_{t+1})}{g'(i_{t+1})}\right)g'(i_t)\right]^{\theta-1} \tag{6.4}$$

$$r_t^f = \frac{1}{E_t(SDF_{t+1})} \tag{6.5}$$

$$R_{t+1} = g'(i_t)\left(A_{t+1}-i_{t+1}+\frac{g(i_{t+1})}{g'(i_{t+1})}\right). \tag{6.6}$$

For convenience of later reference, I define a functional representation of the variables as $SDF_{t+1} = SDF(i_t,\ i_{t+1})$, $r_t^f = r^f(i_t,\ i_{t+1})$, and $R_{t+1} = R(i_t,\ i_{t+1})$.

Proof. See Appendix 9.14.

Unfortunately, the stochastic difference equation (6.3) does not have a general solution, as we had in the baseline model. However, the solution for the deterministic BGP is still available.

6.1 Deterministic BGPs

In this subsection, I examine the deterministic growth path. Ignoring the stochastic components and fixing the productivity parameter A, Proposition 7 offers the following condition, which allows for a BGP in the economy.

Proposition 7. *According to the stochastic difference equation* (6.3), *the deterministic BGP should be at the level of* $i = \bar{i}$, *which solves the following equation:*

$$\frac{(A-i)\,g'(i)+g(i)}{g(i)^{\rho}}=\frac{1}{\beta}. \tag{6.7}$$

Appendix 9.15 discusses the root of this equation. Additionally, in this BGP, the transversality condition holds (see Appendix 9.17). Furthermore, the marginal effect of the technology A-to-investment-to-capital ratio is derived using the implicit function theorem.

$$\frac{\partial i}{\partial A}=\left[(A-i)\left(\rho\frac{g'(i)}{g(i)}-\frac{g''(i)}{g'(i)}\right)+\rho\right]^{-1}>0 \tag{6.8}$$

$$\frac{\partial^2 i}{\partial A^2}=-\left(\rho\frac{g'(i)}{g(i)}-\frac{g''(i)}{g'(i)}\right)\left[(A-i)\left(\rho\frac{g'(i)}{g(i)}-\frac{g''(i)}{g'(i)}\right)+\rho\right]^{-2}<0. \tag{6.9}$$

Proof. In the deterministic case, where the productivity parameter A is a constant, according to the difference equation (6.3), the economy obtains balanced growth when

$$\left[\beta\left(\frac{A-i}{A-i}\cdot g(i)\right)^{-\rho}\left(A-i+\frac{g(i)}{g'(i)}\right)g'(i)\right]^{\theta}=1. \tag{6.10}$$

Basic algebra yields

$$\frac{(A-i)\,g'(i)+g(i)}{g(i)^{\rho}}=\frac{1}{\beta}. \tag{6.11}$$

I directly apply the single variable implicit function theorem to obtain the derivatives. The signs of the derivatives are implied by the assumptions of $g'(i)>0$ and $g''(i)<0$.

Q.E.D. ■

The logic here is similar to that in the baseline model. A static comparison shows that the firm raises its investment to catch up with productivity. This is optimal because it increases future output and the present value of the dividend cash flow. However, there are trade-offs involved. The first trade-off comes from the adjustment cost of the investment. As investment increases, the investment itself becomes less and less efficient, which harms the potential growth of further output. The second trade-off is the typical story of the asset pricing mechanism. The substitution effect conflicts with the income effects. Hence, a high dividend growth alone does not guarantee the high price of the asset because it means low marginal utility in the future.

Accordingly, the expressions of the other variables on this BGP are derived in the following proposition.

Proposition 8. *Owing to the solution of the investment-to-capital ratio $\bar{i}$ to equation* (6.7)*, the following economic variables remain constant at this BGP:*

$$\frac{C_{t+1}}{C_t} = g\left(\bar{i}\right) \tag{6.12}$$

$$r_{t+1}^f = \frac{1}{DF_{t+1}} = \frac{g\left(\bar{i}\right)^\rho}{\beta}, \tag{6.13}$$

where r^f is the risk-free rate.

Proof. See Appendix 9.16.

Furthermore, the stability or local determinacy of the growth path is of interest. Hence, I present Proposition 9 to show that the BGP defined in equation (6.7) is locally unstable.

Proposition 9. *According to the Euler difference equation* (6.3)*, I conclude that the BGP defined in Proposition 7 is locally unstable. If we initialize the economy in the local area of the BGP, the economy diverges.*

Proof. I apply the implicit function theorem on the difference equation (6.3) and yield:

$$\begin{aligned} \frac{\partial i_{t+1}}{\partial i_t} = & \frac{g\left(i_{t+1}\right) + \left(A_{t+1} - i_{t+1}\right) g'\left(i_{t+1}\right)}{g\left(i_t\right)} \times \\ & \frac{A_{t+1} - i_{t+1}}{A_t - i_t} \times \frac{g'\left(i_{t+1}\right)}{g'\left(i_t\right)} \times \\ & \frac{\left(A_t - i_t\right) \rho g'\left(i_t\right)^2 + g\left(i_t\right)\left(\rho g'\left(i_t\right) - \left(A_t - i_t\right) g''\left(i_t\right)\right)}{\left(A_{t+1} - i_{t+1}\right) \rho g'\left(i_{t+1}\right)^2 + g\left(i_{t+1}\right)\left(\rho g'\left(i_{t+1}\right) - \left(A_{t+1} - i_{t+1}\right) g''\left(i_{t+1}\right)\right)}. \end{aligned} \tag{6.14}$$

Thus,

$$\left.\frac{\partial i_{t+1}}{\partial i_t}\right|_{\substack{i_{t+1}\to i_t\to\bar{i}\\ A_{t+1}\to A_t\to A}} = 1+\frac{\left(A-\bar{i}\right)g'\left(\bar{i}\right)}{g\left(\bar{i}\right)} > 1 \tag{6.15}$$

Therefore, in the local area around the BGP, the paths are unstable.

Q.E.D. ■

Owing to the on-linear structure of the difference equation (6.3), it is rather difficult to derive the analytical solution and to examine all the paths. However, because the aim is to study BGPs, I ignore the unstable equilibrium paths. Given the above properties in the BGP, the model is ready to accommodate multiplicities.

The baseline model is fully developed in the EZ framework. In general, the theoretical model re-exhibits the properties shown in the log utility. However, the recursive structure and the separation of IES $1/\rho$ from the risk aversion γ make the solutions less elegant and less intuitive. The model with the log utility has trade-offs in the calibration and cannot capture the macro-fundamental variables and the financial variables simultaneously. In the next section, the model in the EZ framework is calibrated in two alternative ways to overcome these problems.

7 Calibration of the Updated Model

In this section, the model is calibrated to compare the predictions with the data moments in the United States. Table 1 shows the 10 year mean of indicated variables in the United States. Data are collected from two different decades. In contrast to the method used there, this strategy employs the moments presented in Table 1 in the model to determine whether it can find reasonable values for the parameters that coordinate with the data moments. In particular, the calibration in this section no longer takes a specific functional form for the efficiency function g. I only calibrate the model for two equilibria. For the growth rate $g\left(i\right)$, the average growth rates from the data are available. Hence, we simply need to calibrate the marginal growth rate $g'\left(i\right)$.

The model predicts a branch of analytical solutions for the above moments, re-

Variables	$i = I/K$	I/Y	Growth	r^f	$E(R) - r^f$
1992 - 2001 (High Growth)	11.5%	34.8%	2.28%	3.58%	2.4%
2005 - 2014 (Low Growth)	10.4%	32%	0.65%	1.1%	5.6%

Table 1: Data Moments

gardless of the BGPs. In fact, the model yields the theoretical moments for

$$\frac{I}{Y} = \frac{I/K}{AK/K} = \frac{i}{A} \tag{7.1}$$

$$\frac{Y_{t+1}}{Y_t} = \frac{K_{t+1}}{K_t} = g\left(\bar{i}\right) \tag{7.2}$$

$$r^f_{t+1} = \frac{1}{DF_{t+1}} = \frac{g\left(\bar{i}\right)^{\rho}}{\beta}. \tag{7.3}$$

On the other hand, I substitute the collected data moments into the model and assume

$$i_L = 0.104, \qquad i_H = 0.115 \tag{7.4}$$

$$\frac{i_L}{A_L} = 0.32, \qquad \frac{i_H}{A_H} = 0.348 \tag{7.5}$$

$$g\left(i_L\right) = 1.0065, \qquad g\left(i_H\right) = 1.0228 \tag{7.6}$$

$$r^f_L = 1.011, \qquad r^f_H = 1.0358 \tag{7.7}$$

$$E\left(R_L\right) = 1.067, \qquad E\left(R_H\right) = 1.0698 \tag{7.8}$$

Additionally, for convenience of later calculation, the first two rows simply state that

$$A_L = 0.325, \qquad A_H = 0.3305 \tag{7.9}$$

The experiments reflect two scenarios. The first is a simple one. The model is calibrated in two separate BGPs. Then, I construct the regime-switching model and examine the corresponding parameters yielded by the model.

7.1 Two Separate BGPs

This section calibrates the updated model in two separate BGPs. In a sense, this is similar to the situation examined in the calibration of the baseline model. The investment-to-capital ratio i for the BGPs are determined independently and are irrelevant to the distribution of the Markovian sunspots. Proposition 8 proposes a solution for the risk-free rate. For two separate BGPs, we obtain the value parameters for ρ and β from the solution of the following equation system:

$$\frac{g(i_L)^\rho}{\beta} = r_L^f \tag{7.10}$$

$$\frac{g(i_H)^\rho}{\beta} = r_H^f. \tag{7.11}$$

This equation system can be solved for $\rho = 1.51$ and $\beta = 0.998$. Second, the equation (6.7) system is rearranged to

$$(A_L - i_L)\, g'(i_L) + g(i_L) = r_L^f \tag{7.12}$$

$$(A_H - i_H)\, g'(i_H) + g(i_H) = r_H^f. \tag{7.13}$$

The equation system receives the values from data moments, except for $g'(i_L)$ and $g'(i_H)$. Hence, it can be solved for $g'(i_L) = 0.02$ and $g'(i_H) = 0.06$.

Because the data show that $\bar{i}_L = 0.104$ and $\bar{i}_H = 0.115$, the calibration results are contrary to our assumption in the theoretical model that $g''(i) < 0$. I discuss this in next subsection. In fact, it is unreasonable to assume that the data moments are generated by two unrelated, separate BGPs. However, the exercises in this subsection propose anchors for the parameter values of β, ρ, $g'(i_L)$, and $g'(i_H)$.

7.2 Regime Switches

This subsection calibrates the model using regime-switching sunspots, following Assumption 5. Again, the calibration starts with the key stochastic difference Euler equation (6.3):

$$E_t\left\{\left[\beta\left(\frac{A_{t+1} - i_{t+1}}{A_t - i_t}\cdot g(i_t)\right)^{-\rho}\left(A_{t+1} - i_{t+1} + \frac{g(i_{t+1})}{g'(i_{t+1})}\right) g'(i_t)\right]^\theta\right\} = 1. \tag{7.14}$$

The functional expression suggested in Proposition 6 has the reduced form given by

$$E_t\left[SDF(A_t,\ A_{t+1},\ i_t,\ i_{t+1})\, R(A_{t+1},\ i_t,\ i_{t+1})\right] = 1. \tag{7.15}$$

Once again, I substitute all historical data moments back into the model to determine whether the model has reasonable parameters to coordinate these moments. According to the probability distribution of the regime-switching framework, the economy faces the following difference equation system:

$$1 = p_L SDF\left(A_L,\ A_L,\ i_L,\ i_L\right) R\left(A_L,\ i_L,\ i_L\right) + \left(1-p_L\right) SDF\left(A_L,\ A_H,\ i_L,\ i_H\right) R\left(A_H,\ i_L,\ i_H\right) \tag{7.16}$$

$$1 = p_H SDF\left(A_H,\ A_H,\ i_H,\ i_H\right) R\left(A_H,\ i_H,\ i_H\right) + \left(1-p_H\right) SDF\left(A_H,\ A_L,\ i_H,\ i_L\right) R\left(A_L,\ i_H,\ i_L\right). \tag{7.17}$$

These two equations represent that the current state is a slow growth state or a high growth state, respectively. In total, there are seven unknowns in the equations: β, ρ, γ, p_L, p_H, $g'\left(i_L\right)$, and $g'\left(i_H\right)$.

Additionally, I decompose the Euler equation to identify places for the risk-free rate r^f and the expected risky assets returns $E\left(R\right)$ of the data moments. In fact, the expectation of the product of two terms has the following feature:

$$1 = E_t\Big[SDF\left(A_t,\ A_{t+1},\ i_t,\ i_{t+1}\right) R\left(A_{t+1},\ i_t,\ i_{t+1}\right)\Big] \tag{7.18}$$

$$= E_t\Big[SDF\left(A_t,\ A_{t+1},\ i_t,\ i_{t+1}\right)\Big] E_t\Big[R\left(A_{t+1},\ i_t,\ i_{t+1}\right)\Big] + Cov\Big[SDF\left(A_t,\ A_{t+1},\ i_t,\ i_{t+1}\right),\ R\left(A_{t+1},\ i_t,\ i_{t+1}\right)\Big]. \tag{7.19}$$

Therefore, the first two expectation terms in equation (7.19) accommodate r^f_L, r^f_H, $E\left(R_L\right)$, and $E\left(R_H\right)$. For example, from the definition of the risk-free rate in Proposition 6 and the definition of covariance, the model in the low growth state should follow

$$1 = \frac{E\left(R_L\right)}{r^f_L} + E_t\left\{\left[SDF\left(A_t,\ A_{t+1},\ i_t,\ i_{t+1}\right) - \frac{1}{r^f_L}\right]\left[R\left(A_{t+1},\ i_t,\ i_{t+1}\right) - E\left(R_L\right)\right]\right\}. \tag{7.20}$$

Furthermore, the expectation in the second term can be calculated using the probabilities p_L and p_H. In all, the decomposition of the Euler difference equation system proposes the following:

$$1 = \frac{E\left(R_L\right)}{r_L^f} + p_L \left[SDF\left(A_L,\ A_L,\ i_L,\ i_L\right) - \frac{1}{r_L^f}\right]\left[R\left(A_L,\ i_L,\ i_L\right) - E\left(R_L\right)\right] +$$

$$\left(1 - p_L\right)\left[SDF\left(A_L,\ A_H,\ i_L,\ i_H\right) - \frac{1}{r_L^f}\right]\left[R\left(A_H,\ i_L,\ i_H\right) - E\left(R_L\right)\right] \tag{7.21}$$

$$1 = \frac{E\left(R_H\right)}{r_H^f} + p_H \left[SDF\left(A_H,\ A_H,\ i_H,\ i_H\right) - \frac{1}{r_H^f}\right]\left[R\left(A_H,\ i_H,\ i_H\right) - E\left(R_H\right)\right] +$$

$$\left(1 - p_H\right)\left[SDF\left(A_H,\ A_L,\ i_H,\ i_L\right) - \frac{1}{r_H^f}\right]\left[R\left(A_L,\ i_H,\ i_L\right) - E\left(R_H\right)\right]. \tag{7.22}$$

Once again, this is an equation system for unknowns of β, ρ, γ, p_L, p_H, $g'\left(i_L\right)$, and $g'\left(i_H\right)$. Together with the previous two equations (equation (7.16) and (7.17)), we have an equation system with four equations and seven unknowns. Ideally, if we calibrate three of them, say β, ρ, and γ, we can solve for the remaining variables.

Unfortunately, the equation system is highly nonlinear and has no closed-form solution. In fact, numerical methods provided by computational software also have difficulties deriving a numerical solution. Therefore, I resort to minimizing the loss function. The loss function is simply set as the sum of the square of the difference in the four equations. To simplify the notation, Υ_1, Υ_2, Υ_3, and Υ_4 are used to denote the right-hand sides of equations (7.16), (7.17), (7.21), and (7.22), respectively, in the following loss function:

$$LF\left(\eta\right) = \sum_{n=1}^{4}\left[1 - \Upsilon_n\left(\eta\right)\right]^2, \tag{7.23}$$

where η is a vector of unknowns, namely, $\left(\beta, \gamma, \rho, p_L,\ p_H,\ g'\left(i_L\right),\ g'\left(i_H\right)\right)$.

To save computational power, I calibrate the risk-aversion parameter $\gamma = 7$, following the suggestion in Bansal & Yaron (2004). From the standard numerical minimization method in the computational software, when $\beta = 0.979$, $\rho = 0.11$, $p_L = 0.952$, $p_H = 0.999$, $g'\left(i_L\right) = 0.034$, and $g'\left(i_H\right) = 0.016$, the loss function LF is minimized to 2.7×10^{-11}. At the same time, we have $\Upsilon_1 = 0.9994$, $\Upsilon_2 = 0.9990$, $\Upsilon_3 = 1.00004$, and $\Upsilon_4 = 1.00169$. Because the purpose of these experiments is to conduct a calibration, I conclude that the above values are fairly close and reasonable.

With these parameterizations, the model matches the predictions with the data moments well.

After considering all of the data moments, the first feature of the regime-switching model is that it needs relatively persistent parameters of $p_L = 0.952$ and $p_H = 0.999$ to match the data moments. With probability close to one, the economy tends to stay in the current state. If the current state is the slow growth state, the economy tends to be trapped for a long period.

Second, the derivatives of the efficiency function, $g'(i_L) = 0.034$ and $g'(i_H) = 0.016$, are different from the values obtained in the calibrations of the two separate BGPs, namely, $g'(i_L) = 0.02$ and $g'(i_H) = 0.06$. In fact, the former still follows the assumption $g''(i) < 0$. The literature offers no consensus on the proper curvature of the production function $g(i)$ of the capital goods, in reality. Many researchers assume that the adjustment cost for investment is convex, which means the production function of the capital goods should be concave. However, we also can justify the convex production function of capital goods using complementarities or externalities. Though the model proposed here is not capable of judging these two, the second case is clearly empirically more plausible because it can simultaneously capture all 10 moments from the data.

Third, the parameter of 1/IES, $\rho = 0.11$, is significantly different from the value $\rho = 1.51$ in the two separate models. In the literature, the value of IES is controversial. Estimations from Hall (1988) suggest low values for IES of around 0.5, which makes ρ around 2. However, Bansal & Yaron (2004) and van Binsbergen et al. (2012) updated the value to a level to be greater than one. Bansal & Yaron (2004) suggest that the estimation of IES should be modified to around 1.5 when taking into account the effects of time-varying consumption volatility. Hence, the parameter ρ is around 0.66. Given the complex structures in the calibrations, the model is not able to justify the values in the calibration. However, according to the literature, I believe that the value obtained is reasonable.

8 Concluding Remarks

This study explores secular stagnation and its implications for asset pricing. I develop an AK model. With the log utility and the AK production function, the baseline model reveals a linkage between technology and investment. Productivity determines firms' willingness to invest. In addition, Proposition 3 shows that a significant increase in technology A does not necessarily mean a strong improvement in economic growth. Furthermore, the baseline model shows that, with exogenous i.i.d. technology, we have counter-cyclical risk premiums.

Further, I endogenize the technology A. By doing so, the model exhibits multiple equilibria with different economic growth rates, risk-free rates, risk returns, and so forth. I introduce Markovian regime switching sunspots, which serve as a selecting device. The sunspots represent beliefs and activate the switch between BGPs in the economy. In general, the model produces arbitrarily long periods of low growth.

The model has a simple structure and closed-form solutions for all moments of the macroeconomic variables. In the calibration, the model captures the growth rate and risk premium relatively well, but has a trade-off when considering the investment-to-capital ratio. Therefore, the Epstein and Zin framework is introduced to replace the log-utility and to improve the calibration. This framework gives the degree of freedom on the intertemporal elasticity of substitution (IES). Therefore, the model exhibits greater explanatory power on the data.

In the calibration of the updated model, whether the agents are aware of the regime switches and the corresponding probability distribution is important. The calibration of the regime-switching model is empirically more plausible than the calibration of the two separate BGPs. With reasonable values of parameters, the former is capable of accommodating data moments from the macro fundamentals, such as the growth rate, investment-to-capital ratio, investment GDP ratio, as well as from the asset pricing side (the risk-free rate and risk premium) from the two periods.

This study builds a simple-structured endogenous growth model with multiplicity and a regime-switching structure, preliminarily attempting to bring this theoretical framework into the data. Future work should focus on conducting empirical studies. For example, although the nonlinear structure of the Euler difference equation makes the estimation difficult, we still have many methods available in the field of Bayesian estimation that can deal with nonlinear dynamic stochastic general equilibrium models.

9 Appendix

9.1 Convexity of Investment adjustment costs

Here I show that our restrictions on the capital accumulation function are consistent with the convex restriction on the adjustment costs function in the literature. Our capital accumulation condition (3.2) can be re-expressed as

$$K_{t+1} = I_t - [i_t - g(i_t)] K_t \tag{9.1}$$

If the third term is zero, the current period investment is directly transformed into tomorrow's capital input without cost. Accordingly, I can treat the term in the bracket as adjustment costs of investment. Denote it as $Cst(i)$. Then our restrictions $1 > g'(i) > 0$ and $g''(i) < 0$ immediately lead to,

$$Cst'(i) = 1 - g'(i) > 0 \tag{9.2}$$

$$Cst''(i) = -g''(i) > 0 \tag{9.3}$$

i.e. our restrictions on $g(i)$ indicate adjustment costs is convex.

9.2 FOCs and Euler Equation of the Firm's Problem

This section I offer the derivation of the firm's problem in the baseline model.

Proof. The firms' problem can be written in recursive form. The Bellman equation is

$$V_t = \max_{I_t} \quad A_t K_t - I_t + E_t \left[\beta \frac{\Lambda_{t+1}}{\Lambda_t} V_{t+1} \right] \tag{9.4}$$

The first order condition with respect to investment I is simply calculated by taking derivatives with respect to I and equalling it to 0.

$$E_t \left[\beta \frac{\Lambda_{t+1}}{\Lambda_t} \frac{\partial V_{t+1}}{\partial K_{t+1}} \right] = \frac{1}{g'(i_t)} \tag{9.5}$$

For the FOC for capital K, one can either use the envelop theorem or directly apply the Hayashi (1982) proposition in this model. For a problem like this, I have a condition derived by Hayashi (1982) stating that "marginal q" ($\partial V / \partial K$) equals to the "average Q" (V/K). A rigorous proof is in the appendix 9.3. In fact, this condition simplifies the calculation. Therefore, by allowing of the Hayashi proposition, I divide both side of the Bellman equation (9.4) by K_t to obtain the first order condition for capital as

$$\frac{V_t}{K_t} = A_t - i_t + E_t\left[\beta\frac{\Lambda_{t+1}}{\Lambda_t}\frac{V_{t+1}}{K_{t+1}}\frac{K_{t+1}}{K_t}\right] \tag{9.6}$$

$$\frac{V_t}{K_t} = A_t - i_t + \frac{g(i_t)}{g'(i_t)} \tag{9.7}$$

The second equality is obtain by plugging in the FOC of investment I and capital accumulation condition. Finally, I forward the expression (9.7) for one period and substitute it back into equation (9.5) to achieve the Euler equation.

$$E_t\left\{\beta\frac{\Lambda_{t+1}}{\Lambda_t}\left[A_{t+1} - i_{t+1} + \frac{g(i_{t+1})}{g'(i_{t+1})}\right]\right\} = \frac{1}{g'(i_t)} \tag{9.8}$$

Q.E.D. ■

9.3 Proof of Hayashi Proposition in the Model

Essentially what I need to prove is, in this model with the assumed functional form, $V_{K,t} = V_t/K_t$.
Proof. I start from the FOC for I_t,

$$E_t\left[\beta\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right)V_{K_{t+1}}\right] = \frac{1}{g'(i_t)} \tag{9.9}$$

I use the envelope theorem to derive the FOC of capital K_t,

$$V_{Kt} = A_t + E_t\left[\beta\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right)V_{Kt+1}\right]\left[1 - \delta + g(i_t) - i\phi'(i_t)\right] \tag{9.10}$$

I multiply both sides by K_t, yield

$$V_{Kt}K_t = A_tK_t + E_t\left[\beta\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right)V_{Kt+1}\right]\left[K_t\left((1-\delta)+g\left(i_t\right)\right) - i_tg'\left(i_t\right)K_t\right] \quad (9.11)$$

$$= A_tK_t + E_t\left[\beta\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right)V_{Kt+1}K_{t+1}\right] - E_t\left[\beta\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right)V_{Kt+1}\right]g'\left(i_t\right)I_t \quad (9.12)$$

$$= A_tK_t - I_t + E_t\left[\beta\left(\frac{\Lambda_{t+1}}{\Lambda_t}\right)V_{Kt+1}K_{t+1}\right] \quad (9.13)$$

The third equality I use the FOC for I_t in (9.9). Equation (9.13) can be forward and iterated to obtain,

$$V_{K,t}K_t = E_t\left\{\sum_{\tau=t}^{\infty}\beta^{\tau-t}\left(\frac{\Lambda_\tau}{\Lambda_t}\right)\left(A_\tau K_\tau - I_\tau\right)\right\} \quad (9.14)$$

$$= V_t \quad (9.15)$$

Q.E.D. ■

9.4 Consumer's Problem

The representative consumer faces a standard infinite horizon utility maximization problem given by,

$$J_t = \underset{C}{Max} \quad E_t\left[\sum_{\tau=t}^{\infty}\beta^{\tau-t}U\left(C_t\right)\right] \quad (9.16)$$

subject to budget constrain

$$S_{t+1}P_t = S_t\left(P_t + D_t\right) - C_t \quad (9.17)$$

$$C_t > 0, \quad S_t > 0 \quad (9.18)$$

$$P_t > 0, \quad D_t > 0 \quad (9.19)$$

The Bellman equation can be written as

$$J_t\left(S_t\right) = \underset{C}{Max} \quad U\left(C_t\right) + \beta E_t\left(J_{t+1}\left(S_{t+1}\right)\right) \quad (9.20)$$

FOC of consumption C_t gives,

$$U'(C_t) = \beta E_t \left(\frac{J_{St+1}}{P_t} \right) \tag{9.21}$$

Envelope theorem offers

$$J_{St} = \beta E_t \left[J_{St+1} \left(\frac{P_t + D_t}{P_t} \right) \right] \tag{9.22}$$

Combine the two, yield

$$J_{St} = U'(C_t)(P_t + D_t) \tag{9.23}$$

Further, I forward 1 period and substitute back into (9.22). I obtain

$$U'(C_t)(P_t + D_t) = \beta E_t \left[U'(C_{t+1})(P_{t+1} + D_{t+1}) \left(\frac{P_t + D_t}{P_t} \right) \right] \tag{9.24}$$

$$U'(C_t) P_t = \beta E_t \left[U'(C_{t+1})(P_{t+1} + D_{t+1}) \right] \tag{9.25}$$

which is the Euler equation.

9.5 Transversality Condition of Baseline Model

In the firms problem, the transversality condition is

$$\lim_{\tau \to \infty} \beta^{\tau} \frac{\Lambda_{\tau}}{\Lambda_t} \frac{\partial D_{\tau}}{\partial K_{\tau+1}} K_{\tau+1} = 0 \tag{9.26}$$

$$\lim_{\tau \to \infty} \beta^{\tau} \frac{\Lambda_{\tau}}{\Lambda_t} g'(i_{\tau})^{-1} K_{\tau+1} = 0 \tag{9.27}$$

$$\frac{1}{\Lambda_t} \lim_{\tau \to \infty} \beta^{\tau} \frac{K_{\tau+1}}{A_{\tau} K_{\tau} - I_{\tau}} g'(i_{\tau})^{-1} = 0 \tag{9.28}$$

$$\frac{1}{\Lambda_t} \lim_{\tau \to \infty} \beta^{\tau} \frac{g(i_{\tau})}{(A - i_{\tau})\, g'(i_{\tau})} = 0 \tag{9.29}$$

Since i is a constant in the equilibrium, the condition is satisfied.

The consumer's problem has a transversality condition given by

$$\lim_{t \to \infty} \beta^{t} U'(C_t) P_t = 0$$

In the equilibrium, consumption constantly grow at rate $g(i)$, given C_t

$$\begin{aligned}\lim_{t\to\infty}\beta^t U'(C_t)P_t &= \lim_{t\to\infty}\beta^t\frac{P_t}{D_t} && (9.30)\\ &= \lim_{t\to\infty}\beta^t\frac{V_t-D_t}{D_t} && (9.31)\\ &= \lim_{t\to\infty}\beta^t\left(\frac{V_t/K_t}{A_t-i_t}-1\right) && (9.32)\\ &= \lim_{t\to\infty}\beta^t\left(\frac{A_t-i_t+\frac{g(i_t)}{g'(i_t)}}{A_t-i_t}-1\right) && (9.33)\\ &= \lim_{t\to\infty}\beta^t\left(\frac{g(i_t)}{(A_t-i_t)g'(i_t)}\right) && (9.34)\end{aligned}$$

In order to follow the typical notation of Euler equation, our asset price P in the consumer's problem is slightly different from the stock price V in the firm's problem. In fact, I have $P+D=V$, which is used in the second equality.

9.6 Proposition 2

Firstly I rearrange the difference equation (3.14) into,

$$\frac{g(i_{t+1})}{(A-i_{t+1})g'(i_{t+1})}=\frac{1}{\beta}\frac{g(i_t)}{(A-i_t)g'(i_t)}-1 \tag{9.35}$$

An change of variable makes the difference equation clearer. I define

$$F_t(i_t)\equiv\frac{g(i_t)}{(A-i_t)g'(i_t)} \tag{9.36}$$

Immediately, the difference equation (9.35) is

$$F_{t+1}=\frac{1}{\beta}F_t-1 \tag{9.37}$$

Next, I show that the F function is monotonically increasing with investment to capital ratio i. Once this relation is established, the solution of the difference equation (9.37) can be generalised to difference equation (9.35).

It is straight forward that

$$F'(i) = \frac{g'(i)^2 (A-i) + [g'(i) - g''(i)(A-i)][g(i)]}{[(A-i) g'(i)]^2} > 0 \tag{9.38}$$

where I applied the assumptions that (1) $D > 0$ therefore $A - i > 0$, (2) $g(i) > 0$, (3) $g'(i) > 0$, (4) $1 - \delta > 0$ and (5) $g''(i) < 0$.

Finally, with the first condition (3.16) in the proposition, the solutions of the difference equation (9.37) have three kinds of potential paths.

1. If $F_0 < \beta/(1-\beta)$ and correspondingly $i_0 < \bar{i}$, then F_t and i_t decrease over time and eventually break the assumption of $i > 0$.

2. If $F_0 > \beta/(1-\beta)$ and correspondingly $i_0 > \bar{i}$, then F_t and i_t increase over time and eventually break the transversality condition. This is shown by the general solution of the difference equation (9.37).

$$F_t = \frac{1}{\beta^t} F_0 - \frac{\beta}{\beta - 1}\left(1 - \frac{1}{\beta^t}\right) \tag{9.39}$$

$$\lim_{t\to\infty} \beta^t F_t = F_0 - \frac{\beta}{1-\beta} \tag{9.40}$$

The transversality condition in the last equality can not be 0 if $F_0 > \beta/(1-\beta)$.

3. If $F_0 = \beta/(1-\beta)$ and correspondingly $i_0 = \bar{i}$, then F_t and i_t stay at this fixed point meanwhile transversality condition holds.

$$\lim_{t\to\infty} \beta^t F_t = F_0 - \frac{\beta}{1-\beta} = 0 \tag{9.41}$$

Additionally, the variable i is restricted by the range $(0, A)$ since $I > 0$ and $A - i > 0$. Hence, the value of the function $F(i)$ falls into

$$F(i) \in \left(\frac{g(0)}{Ag'(0)}, \infty\right) \tag{9.42}$$

Therefore, the first condition (3.16) in this proposition guarantees that the fixed point $\bar{i}$ is feasible.

In all, in our model, there is no transitional dynamics. The only feasible path is the one that initiates the economy at $i = \bar{i}$.

Q.E.D. ■

9.7 Root of the Solution in the Baseline Model

This section I show there is only root in the model solution equation in the baseline model. It is clear the equation (3.17) is non-linear for a given A in the defined range $A > i > 0$.

$$\frac{g(i)}{g'(i)(A-i)} = \frac{\beta}{1-\beta} \tag{9.43}$$

Firstly, I consider the number of roots to this equation.

With the restriction of $A > i$, $g(i) > 0$, $g'(i) > 0$ and $g''(i) < 0$, I can write

$$(1-\beta) g(i) - \beta\phi'(i)(A-i) = 0 \tag{9.44}$$

If I define the left hand side as $f(i)$, in the defined range $A > i > 0$, I have

$$\frac{\partial f}{\partial i} = (1-\beta) g'(i) - \beta \left[g''(i)(A-i) - g'(i)\right] \tag{9.45}$$

$$\frac{\partial f}{\partial i} > 0 \tag{9.46}$$

Additionally, I have

$$f(0) = (1-\beta) g(0) - \beta\phi'(0) A \tag{9.47}$$

$$f(A) = (1-\beta) g(A) \tag{9.48}$$

Thus the sufficient condition grantees the uniqueness of the root is

$$A > \frac{(1-\beta) g(0)}{\beta\phi'(0)} \tag{9.49}$$

which is identical to the condition in proposition 2.

Secondly, I derive the relation between A and i. Implicit function theorem offers

$$\frac{\partial i}{\partial A} = \frac{\beta\phi'(i)}{g'(i) - \beta(A-i)g''(i)} \tag{9.50}$$

$$\frac{\partial^2 i}{\partial A^2} = \frac{\beta^2 g'(i) g''(i)}{\left[g'(i) - \beta(A-i)g''(i)\right]^2} \tag{9.51}$$

In the first derivative, I have that $\beta\phi'(i) < g'(i)$ and $-\beta(A-i)g''(i) > 0$. With the previous conditions, I have $1 > \partial i/\partial A > 0$ and $\partial^2 i/\partial A^2 < 0$.

Accordingly,

$$\frac{\partial(A-i)}{\partial A} = 1 - \frac{\partial i}{\partial A} > 0 \tag{9.52}$$

Naturally, I have $A_H - i[A_H] > A_L - i[A_L]$.

9.8 The Asset Return and Risk-Free Rate under Sunspots

I recall that under proposition 4 and assumption 1, I have the expressions for asset return and risk-free rate as

$$E_t\left(R_{t+1}\middle| i_t = i_L\right) = \frac{g'(i_L)}{1-\beta} E_t\left[A_{t+1} - i\left[A_{t+1}\right]\middle| i_t = i_L\right] \tag{9.53}$$

$$E_t\left(R_{t+1}\middle| i_t = i_H\right) = \frac{g'(i_H)}{1-\beta} E_t\left[A_{t+1} - i\left[A_{t+1}\right]\middle| i_t = i_H\right] \tag{9.54}$$

$$\left[r_t^f\middle| i_t = i_L\right] = \frac{g'(i_L)}{1-\beta}\left[E\left(\frac{1}{A(s^{t+1}) - i\left[A(s^{t+1})\right]}\middle| i_t = i_L\right)\right]^{-1} \tag{9.55}$$

$$\left[r_t^f\middle| i_t = i_H\right] = \frac{g'(i_H)}{1-\beta}\left[E\left(\frac{1}{A(s^{t+1}) - i\left[A(s^{t+1})\right]}\middle| i_t = i_H\right)\right]^{-1} \tag{9.56}$$

With the sunspots assumption 2, the expectation is calculated simply by adding two realisations with their corresponding possibilities. For example,

$$E_t\left(R_{t+1}\middle| i_t = i_L\right) = \frac{g'(i_L)}{1-\beta} E_t\left[A_{t+1} - i\left[A_{t+1}\right]\middle| i_t = i_L\right] \tag{9.57}$$

$$= \frac{g'(i_L)}{1-\beta}\left[(1-p_L)AiH + p_L AiL\right] \tag{9.58}$$

$$= \frac{g'(i_L)}{1-\beta}\left[AiH - p_L(AiH - AiL)\right] \tag{9.59}$$

where $AiL \equiv A_L - i_L$ and $AiH \equiv A_H - i_H$. The rest is obtained by identical methods.

9.9 Risky Asset Return and risk-free Rate

This section in the Appendix I derive the return for the risky asset R_{t+1} the risk-free rate r_t^f. I use the following conditions derived in the baseline model.

$$K_{t+1} = (1-\delta)\, K_t + g\,(i_t)\, K_t \tag{9.60}$$

$$\frac{V\,(K_t)}{K_t} = A_t - i_t + \frac{g\,(i_t)}{g'\,(i_t)} \tag{9.61}$$

$$\frac{g\,(i_t)}{g'\,(i_t)\,(A_t - i_t)} = \frac{\beta}{1-\beta} \tag{9.62}$$

I start from the included dividends asset return,

$$R_{t+1}^{IND} = \frac{V_{t+1}}{V_t} \tag{9.63}$$

$$= \frac{V_{t+1}/K_{t+1}}{V_t/K_t}\frac{K_{t+1}}{K_t} \tag{9.64}$$

$$= \frac{A_{t+1} - i_{t+1} + \frac{g(i_{t+1})}{g'(i_{t+1})}}{A_t - i_t + \frac{g(i_t)}{g'(i_t)}}\,(g\,(i_t)) \tag{9.65}$$

$$= \frac{A_{t+1} - i_{t+1} + (A_{t+1} - i_{t+1})\,\frac{g(i_{t+1})}{g'(i_{t+1})(A_{t+1} - i_{t+1})}}{A_t - i_t + (A_t - i_t)\,\frac{g(i_t)}{g'(i_t)(A_t - i_t)}}\,(g\,(i_t)) \tag{9.66}$$

$$= \frac{(A_{t+1} - i_{t+1})}{(A_t - i_t)}\,[g\,(i_t)] \tag{9.67}$$

$$= (A_{t+1} - i_{t+1})\, g'\,(i_t)\, \frac{[g\,(i_t)]}{g'\,(i_t)\,(A_t - i_t)} \tag{9.68}$$

$$= (A_{t+1} - i_{t+1})\, g'\,(i_t)\, \frac{\beta}{1-\beta} \tag{9.69}$$

Since V_t is the included dividends price I derive the ex-dividends return as

$$R_{t+1} = \frac{P_{t+1} + D_{t+1}}{P_t} \tag{9.70}$$

$$= \frac{V_{t+1}}{V_t - D_t} \tag{9.71}$$

$$= \left[\frac{1}{R_{t+1}^{IND}} - \frac{D_t}{V_{t+1}}\right]^{-1} \tag{9.72}$$

$$= \left[\frac{1}{R_{t+1}^{IND}} - \frac{D_t/K_t}{V_{t+1}/K_{t+1}} \frac{K_t}{K_{t+1}}\right]^{-1} \tag{9.73}$$

$$= \left[\frac{1}{R_{t+1}^{IND}} - \frac{A_t - i_t}{A_{t+1} - i_{t+1} + \frac{g(i_{t+1})}{g'(i_{t+1})}} \frac{1}{g\left(i_t\right)}\right]^{-1} \tag{9.74}$$

$$= \left[\frac{1}{R_{t+1}^{IND}} - \frac{1}{A_{t+1} - i_{t+1} + \left(A_{t+1} - i_{t+1}\right) \frac{\beta}{1-\beta}} \frac{1}{\left(\frac{\beta}{1-\beta}\right) g'\left(i_t\right)}\right]^{-1} \tag{9.75}$$

$$= \left[\frac{1}{R_{t+1}^{IND}} - \frac{1}{\left(A_{t+1} - i_{t+1}\right) \left(\frac{1}{1-\beta}\right)} \frac{1}{\left(\frac{\beta}{1-\beta}\right) g'\left(i_t\right)}\right]^{-1} \tag{9.76}$$

$$= \left[\frac{1}{\left(A_{t+1} - i_{t+1}\right) g'\left(i_t\right) \frac{\beta}{1-\beta}} - \frac{1}{\left(A_{t+1} - i_{t+1}\right) \left(\frac{1}{1-\beta}\right)} \frac{1}{\left(\frac{\beta}{1-\beta}\right) g'\left(i_t\right)}\right]^{-1} \tag{9.77}$$

$$= \left\{\frac{1}{\left(A_{t+1} - i_{t+1}\right) g'\left(i_t\right) \frac{\beta}{1-\beta}} \left[1 - \frac{1}{\left(\frac{1}{1-\beta}\right)}\right]\right\}^{-1} \tag{9.78}$$

$$= \left(A_{t+1} - i_{t+1}\right) \frac{g'\left(i_t\right)}{1-\beta} \tag{9.79}$$

The capital accumulation condition (9.60) and the solution for marginal q (9.61) are used in equality (9.65). The rest of the equality I repeatedly use the solution condition (9.62) for BGP level of i.

One can check the relation indicated by the Euler equation,

$$1 = E_t \left[\beta \frac{\Lambda_{t+1}}{\Lambda_t} \cdot R_{t+1}\right] \tag{9.80}$$

For the risk-free rate, by definition it is the inverse of expectation of SDF as,

$$
\begin{aligned}
r_t^f &= \frac{1}{E\left(\beta\frac{\Lambda_{t+1}}{\Lambda_t}\right)} && (9.81)\\
&= \left[E\left(\beta\cdot\frac{A_t-i_t}{A_{t+1}-i_{t+1}}\frac{K_t}{K_{t+1}}\right)\right]^{-1} && (9.82)\\
&= \left[E\left(\frac{1}{A_{t+1}-i_{t+1}}\right)\beta\left(\frac{A_t-i_t}{g\left(i_t\right)}\right)\right]^{-1} && (9.83)\\
&= \left[E\left(\frac{1}{A_{t+1}-i_{t+1}}\right)\beta\left(\frac{1-\beta}{\beta\phi'\left(i_t\right)}\right)\right]^{-1} && (9.84)\\
&= \left[E\left(\frac{1}{A_{t+1}-i_{t+1}}\right)\right]^{-1}\frac{g'\left(i_t\right)}{1-\beta} && (9.85)
\end{aligned}
$$

The capital accumulation condition (9.60) is used in the third equality and the solution condition(9.62) for of i is used in the forth equality.

9.10 Calibration of the Baseline Model

This section aims to match the theoretical predicted moments to historical data moments. Firstly, I consider two alternative functional forms for the efficiency function $g\left(i\right)$ of the investment to capital ratio i, which are borrowed from Eberly & Wang (2009) and Gourio (2012). The former is in a log form given by

$$
g\left(i\right)=\alpha+\Gamma\log\left(1+\frac{i}{\theta}\right)+1-\delta \qquad (9.86)
$$

The latter is the frequently used quadratic form

$$
g\left(i\right)=i-\frac{\Gamma\left(i-\theta\right)^2}{2}+1-\delta \qquad (9.87)
$$

where α, Γ and θ determine the shape of the efficiency function. $\delta\in(0,\ 1)$ stands for the capital-depreciation rate.

Now the model is really for calibration. Given A_L, A_H, p_L, p_H, β, δ and parameterisations in the efficiency function $g\left(i\right)$, the closed form solutions for the investment-GDP ratio $I/Y=i/A$, the growth rate of the economy $g\left(i\right)$, the risk-free rate r^f

and the expected risky asset return $E\left(R\right)$ in both states are given by

$$\left[\left.\frac{I_t}{K_t}\right| s_t = s_L\right] = i_L, \qquad \left[\left.\frac{I_t}{K_t}\right| s_t = s_H\right] = i_H \tag{9.88}$$

$$\left[\left.\frac{I_t}{Y_t}\right| s_t = s_L\right] = \frac{i_L}{A_L}, \qquad \left[\left.\frac{I_t}{Y_t}\right| s_t = s_H\right] = \frac{i_H}{A_H} \tag{9.89}$$

$$\left[\left.\frac{Y_{t+1}}{Y_t}\right| s_t = s_L\right] = g\left(i_L\right) \tag{9.90}$$

$$\left[\left.\frac{Y_{t+1}}{Y_t}\right| s_t = s_H\right] = g\left(i_H\right) \tag{9.91}$$

$$\left[\left. r_t^f \right| s_t = s_L\right] = \frac{g'\left(i_L\right)}{1-\beta}\left[\frac{1}{AiH} - p_L\left(\frac{1}{AiH} - \frac{1}{AiL}\right)\right]^{-1} \tag{9.92}$$

$$\left[\left. r_t^f \right| s_t = s_H\right] = \frac{g'\left(i_H\right)}{1-\beta}\left[\frac{1}{AiL} + p_H\left(\frac{1}{AiH} - \frac{1}{AiL}\right)\right]^{-1} \tag{9.93}$$

$$E\left[\left. R_{t+1}\right| s_t = s_L\right] = \frac{g'\left(i_L\right)}{1-\beta}\left[AiH - p_L\left(AiH - AiL\right)\right] \tag{9.94}$$

$$E\left[\left. R_{t+1}\right| s_t = s_H\right] = \frac{g'\left(i_H\right)}{1-\beta}\left[AiL + p_H\left(AiH - AiL\right)\right] \tag{9.95}$$

Additionally, the i_L and i_H are determined within the model. The Euler equation (3.13) in general equilibrium splits into two equations in the economy with sunspots.

$$1 = p_L\left[\beta\frac{\left(A_L - i_L + \frac{g(i_L)}{g'(i_L)}\right)g'\left(i_L\right)}{\frac{A_L - i_L}{A_L - i_L}g\left(i_L\right)}\right] + \left(1 - p_L\right)\left[\beta\frac{\left(A_H - i_H + \frac{g(i_H)}{g'(i_H)}\right)g'\left(i_L\right)}{\frac{A_H - i_H}{A_L - i_L}g\left(i_L\right)}\right] \tag{9.96}$$

$$1 = p_H\left[\beta\frac{\left(A_H - i_H + \frac{g(i_H)}{g'(i_H)}\right)g'\left(i_H\right)}{\frac{A_H - i_H}{A_H - i_H}g\left(i_H\right)}\right] + \left(1 - p_H\right)\left[\beta\frac{\left(A_L - i_L + \frac{g(i_L)}{g'(i_L)}\right)g'\left(i_H\right)}{\frac{A_L - i_L}{A_H - i_H}g\left(i_H\right)}\right] \tag{9.97}$$

Ideally, the calibration procedure should firstly solve this equation system for i_L and i_H. Then, we substitute them back into the equations (9.88) to (9.95) to obtain their values. Finally, by varying the parameters, the model obtains different predictions for the moments. In fact, with the condition $p_H \in (0,\, 1)$ and $p_L \in (0,\, 1)$, the above

	α	Γ	θ	δ	β	A_L	A_H	p_L	p_H
Log	0.1	0.015	0.035	0.13	0.98	0.171	2.05	0.9935	0.998
Qud	–	1.7	0.025	0.13	0.98	0.11	0.14	0.9983	0.998

Table 2: Parameterisation

equation system collapses to

$$\frac{g\left(i_H\right)}{g'\left(i_H\right)\left(A_H - i_H\right)} = \frac{\beta}{1-\beta} \tag{9.98}$$

$$\frac{g\left(i_L\right)}{g'\left(i_L\right)\left(A_L - i_L\right)} = \frac{\beta}{1-\beta} \tag{9.99}$$

The calculation is in appendix 9.11. Basically, this means that, with or without the information of the transitional matrix, the investor makes same decision on the level of BGP.

In terms of the parameterisations, the benchmark of the parameters in the efficiency function $g\left(i\right)$ is based on the paper by Eberly & Wang (2009) and Gourio (2012). The others are adjusted to match the model predictions to data moments. The table 2 reports the parameterisations chosen in end.

The moments generated by the model and calculated from the data are in table 3. I use the data of the US including investment-capital ratio, investment-GDP ratio, per capita GDP growth rate, treasury bond rate, and equity risk premium. All data are in real terms. Data description is in appendix 9.12. Particularly, I use the period between 1992 to 2001 as the representation of high-growth state with a 2.28% per capita GDP growth. For the low-growth state, I use 2005 to 2014. In this period, on average, the growth rate of per capita GDP is 0.65%.

As shown in table 3, overall, the investment adjustment function in the log form slightly performs better than the the quadratic from. Figures in bold are those moments cannot match the data moments even in a rough sense. In general, the log-form model can match the moments of GDP growth rate and counter-cyclical risk premium in both states. Especially, it captures the property that in the low-growth state risk-free rate drop dramatically to 0.9%. Meanwhile, it yields a high risk premium of 5.5%. However, there is a trade-off in the calibration. The log-form can not obtain proper investment-capital ratio. The 84% of i_H shows that the model needs a very high $A_H = 2.05$ to generate high growth. In terms of the quadratic

%	Calibration - Log		Calibration - Qud		Data of US	
	High	Low	High	Low	High	Low
$i = I/K$	**84**	5.3	17.5	12.7	11.5	10.4
I/Y	40.9	31.0	**86.1**	**83.8**	34.8	32
Growth Rate	1.8	**-1.6**	2.5	**-1.1**	2.28	0.65
risk-free Rate	**2.0**	0.9	4.6	0.85	3.58	1.1
Risk Premium	**1.7**	5.5	**0.00**	**0.00**	2.4	5.6

Table 3: Calibration and Data

form, it obtains the proper values for investment-capital ratio at around 10% and well performed risk-free rates of $r_H^f = 4.6\%$ and $r_L^f = 0.85\%$. Nonetheless, it cannot offer an appropriate distance between technology A and i to generate the investment-GDP ratio I/Y. In addition, although it generates counter-cyclical risk premium, they are negligible.

Generally, the calibration shows the model can capture some patterns shown in the data. However, there are trade-offs. Those parameters that can generate correct macro-fundamental moments cannot generate reasonable moments of financial variables and vice versa. To overcome this problem, section 6 introduces Epstein and Zin utility to call for more parameters and study influence of the intertemporal elasticity of substitution (IES). There are many studies show that the recursive utility can provide some theoretical explanations of the behaviour of the risk-free rate. Introducing this preference might also improve the calibration of this model.

9.11 Calculation of the Equation System in the Calibration

Basically, I have the equation system

$$p_L \left[\beta \left(\frac{A_L - i_L}{A_L - i_L} \left(g \left(i_L \right) \right) \right)^{-1} \left(A_L - i_L + \frac{g \left(i_L \right)}{g' \left(i_L \right)} \right) g' \left(i_L \right) \right] + \tag{9.100}$$

$$(1 - p_L) \left[\beta \left(\frac{A_H - i_H}{A_L - i_L} \left(g \left(i_L \right) \right) \right)^{-1} \left(A_H - i_H + \frac{g \left(i_H \right)}{g' \left(i_H \right)} \right) g' \left(i_L \right) \right] = 1$$

$$p_H \left[\beta \left(\frac{A_H - i_H}{A_H - i_H} \left(g \left(i_H \right) \right) \right)^{-1} \left(A_L - i_L + \frac{g \left(i_H \right)}{g' \left(i_H \right)} \right) g' \left(i_H \right) \right] + \tag{9.101}$$

$$(1 - p_H) \left[\beta \left(\frac{A_L - i_L}{A_H - i_H} \left(g \left(i_H \right) \right) \right)^{-1} \left(A_L - i_L + \frac{g \left(i_L \right)}{g' \left(i_L \right)} \right) g' \left(i_H \right) \right] = 1$$

To simplify the notation, I rearrange it into

$$p_L \beta \left(\frac{1}{F_L} + 1 \right) + (1 - p_L) \frac{\beta}{F_L} (1 + F_H) = 1 \tag{9.102}$$

$$p_H \beta \left(\frac{1}{F_H} + 1 \right) + (1 - p_H) \frac{\beta}{F_H} (1 + F_L) = 1 \tag{9.103}$$

where the notation of F follows the proposition 2.

$$F_H = \frac{g \left(i_H \right)}{g' \left(i_H \right) \left(A_H - i_H \right)} \tag{9.104}$$

$$F_L = \frac{g \left(i_L \right)}{g' \left(i_L \right) \left(A_L - i_L \right)} \tag{9.105}$$

Further, after some basic algebra, with the conditions that $\beta \neq 0$, $p_L \in (0, 1)$ and $p_H \in (0, 1)$, the equation system has and only have one solution,

$$F_H = \frac{\beta}{1 - \beta} \tag{9.106}$$

$$F_L = \frac{\beta}{1 - \beta} \tag{9.107}$$

Since I have proofed the one to one relation between i and F in proposition 2, the conclusion follows.

9.12 Data

I use the 1992 - 2001 as the high growth state and 2005 to 2014 as the low growth state. Data come from Fred Economic Data[8] and Quandl[9]. All data are quarterly data and adjusted by CPI growth rate. GDP growths are in per capita form. The Risk premium is calculated by the real return on stock index S&P 500 neglecting the real 10-year treasury constant maturity rate.

9.13 Derivation of Discount Factor in the Epstein and Zin Recursive Utility Framework

I start with the recursive value function for the consumer,

$$J_t = \left\{ C_t^{1-\rho} + \beta \left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1-\rho}{1-\gamma}} \right\}^{\frac{1}{1-\rho}} \tag{9.108}$$

Since the value function J_t is homogeneous of degree one, I can rewrite the function with Euler's theorem

$$J_t = \frac{\partial J_t}{\partial C_t} \cdot C_t + E_t \left(\frac{\partial J_t}{\partial J_{t+1}} \cdot J_{t+1} \right) \tag{9.109}$$

The partial derivatives can be derived as

$$\frac{\partial J_t}{\partial C_t} = J_t^{\rho} C_t^{-\rho} \tag{9.110}$$

$$\frac{\partial J_t}{\partial \left(\left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1}{1-\gamma}} \right)} = J_t^{\rho} \beta \left(\left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1}{1-\gamma}} \right)^{-\rho} \tag{9.111}$$

$$\frac{\partial \left(\left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1}{1-\gamma}} \right)}{\partial J_{t+1}} = \left(\left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{\gamma}{1-\gamma}} \right) J_{t+1}^{-\gamma} \tag{9.112}$$

$$\frac{\partial J_t}{\partial J_{t+1}} = \frac{\partial J_t}{\partial \left(\left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1}{1-\gamma}} \right)} \frac{\partial \left(\left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1}{1-\gamma}} \right)}{\partial J_{t+1}} \tag{9.113}$$

$$= J_t^{\rho} \beta \left(\left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1}{1-\gamma}} \right)^{\gamma-\rho} J_{t+1}^{-\gamma} \tag{9.114}$$

[8] research.stlouisfed.org

[9] www.quandl.com/data/MULTPL/SP500_INFLADJ_MONTH

Hence, the intertemporal marginal rate of substitution is given by

$$IMRS_{t+1} = \frac{\frac{\partial J_t}{\partial J_{t+1}} \frac{\partial J_{t+1}}{\partial C_{t+1}}}{\frac{\partial J_t}{\partial C_t}} \tag{9.115}$$

$$= \frac{J_t^{\rho} \beta \left(\left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1}{1-\gamma}} \right)^{\gamma-\rho} J_{t+1}^{-\gamma} J_{t+1}^{\rho} C_{t+1}^{-\rho}}{J_t^{\rho} C_t^{-\rho}} \tag{9.116}$$

$$= \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{J_{t+1}}{\left[E_t \left(J_{t+1}^{1-\gamma} \right) \right]^{\frac{1}{1-\gamma}}} \right)^{\rho-\gamma} \tag{9.117}$$

Define the total wealth W_t and the cum-dividend return $R_{w,t+1}$ on wealth to be

$$W_t = C_t + E_t IMRS_{t+1} W_{t+1} \tag{9.118}$$

$$R_{w,t+1} = \frac{W_{t+1}}{W_t - C_t} \tag{9.119}$$

Recall that I have

$$J_t = \frac{\partial J_t}{\partial C_t} \cdot C_t + E_t \left(\frac{\partial J_t}{\partial V_{t+1}} \cdot J_{t+1} \right) \tag{9.120}$$

Therefore,

$$W_t = \frac{J_t}{\partial J_t / \partial C_t} \tag{9.121}$$

Further, the relation between wealth return and $IMRS_{t+1}$ is established as

$$R_{w,t+1} = \frac{\frac{J_{t+1}}{\partial J_{t+1}/\partial C_{t+1}}}{\frac{J_t}{\partial J_t/\partial C_t} - C_t} \tag{9.122}$$

$$= \left(\frac{C_{t+1}}{C_t}\right)^{\rho} \frac{J_{t+1}^{1-\rho}}{J_t^{1-\rho} - C_t^{1-\rho}} \tag{9.123}$$

$$= \frac{1}{\beta}\left(\frac{C_{t+1}}{C_t}\right)^{\rho} \left(\frac{J_{t+1}}{\left[E_t\left(J_{t+1}^{1-\gamma}\right)\right]^{\frac{1}{1-\gamma}}}\right)^{1-\rho} \tag{9.124}$$

$$\frac{J_{t+1}}{\left[E_t\left(J_{t+1}^{1-\gamma}\right)\right]^{\frac{1}{1-\gamma}}} = \left[R_{w,t+1}\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right]^{\frac{1}{1-\rho}} \tag{9.125}$$

$$IMRS_{t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left(\frac{J_{t+1}}{\left[E_t\left(J_{t+1}^{1-\gamma}\right)\right]^{\frac{1}{1-\gamma}}}\right)^{\rho-\gamma} \tag{9.126}$$

$$= \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left[R_{w,t+1}\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right]^{\frac{\rho-\gamma}{1-\rho}} \tag{9.127}$$

$$= \beta^{1+\frac{\rho-\gamma}{1-\rho}} \left(\frac{C_{t+1}}{C_t}\right)^{-\rho\left(1+\frac{\rho-\gamma}{1-\rho}\right)} R_{w,t+1}^{\frac{\rho-\gamma}{1-\rho}} \tag{9.128}$$

$$= \left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right]^{\theta} \left(\frac{1}{R_{w,t+1}}\right)^{1-\theta} \tag{9.129}$$

Where $\theta \equiv (1-\gamma)/(1-\rho)$. In the third equality, I apply the fact that $J_t^{1-\rho} = C_t^{1-\rho} + \beta\left[E_t\left(J_{t+1}^{1-\gamma}\right)\right]^{\frac{1-\rho}{1-\gamma}}$.

9.14 Proposition 6

Following the baseline model, I have optimal condition for the firm's partial equilibrium problem as

$$E_t\left[\beta\frac{\Lambda_{t+1}}{\Lambda_t}\left(A_{t+1} - i_{t+1} + \frac{g(i_{t+1})}{g'(i_{t+1})}\right)g'(i_t)\right] = 1 \tag{9.130}$$

In the standard asset pricing representation, it can be reformed into the form of Euler equation,

$$E_t\left[SDF_{t+1}R_{t+1}\right]=1 \tag{9.131}$$

It is obvious to identify the asset return in the model to be

$$R_{t+1}=\left(A_{t+1}-i_{t+1}+\frac{g\left(i_{t+1}\right)}{g'\left(i_{t+1}\right)}\right)g'\left(i_t\right) \tag{9.132}$$

Recall the discount factor derived from consumer's problem as

$$SDF_{t+1}=\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right]^{\theta}\left(\frac{1}{R_{t+1}}\right)^{1-\theta} \tag{9.133}$$

With all these preparation and the market clearing condition $C=D$, our Euler equation can be rewritten as

$$E_t\left\{\left[\beta\left(\frac{\left(A_{t+1}-i_{t+1}\right)K_{t+1}}{\left(A_t-i_t\right)K_t}\right)^{-\rho}\left(A_{t+1}-i_{t+1}+\frac{g\left(i_{t+1}\right)}{g'\left(i_{t+1}\right)}\right)g'\left(i_t\right)\right]^{\theta}\right\}=1 \tag{9.134}$$

$$E_t\left\{\left[\beta\left(\frac{A_{t+1}-i_{t+1}}{A_t-i_t}\cdot g\left(i_t\right)\right)^{-\rho}\left(A_{t+1}-i_{t+1}+\frac{g\left(i_{t+1}\right)}{g'\left(i_{t+1}\right)}\right)g'\left(i_t\right)\right]^{\theta}\right\}=1 \tag{9.135}$$

By definition, the risk free rate is

$$r_t^f=\frac{1}{E_t\left[DF_{t+1}\right]} \tag{9.136}$$

$$=\left[E_t\left\{\beta^{\theta}\left(\frac{A_{t+1}-i_{t+1}}{A_t-i_t}\cdot g\left(i_t\right)\right)^{-\theta\rho}\left[\left(A_{t+1}-i_{t+1}+\frac{g\left(i_{t+1}\right)}{g'\left(i_{t+1}\right)}\right)g'\left(i_t\right)\right]^{\theta-1}\right\}\right]^{-1} \tag{9.137}$$

For the price dividend ratio and decomposition of the asset return, I firstly derive the average price of capital V_t/K_t as a corner stone. The FOC for the firm's problem is

$$E_t\left[\beta\frac{\Lambda_{t+1}}{\Lambda_t}\frac{\partial V_{t+1}}{\partial K_{t+1}}g'\left(i_t\right)\right]=1 \tag{9.138}$$

Hayashi (1982) has the proposition enables us to rewrite the bellman equation.

$$\frac{V_t}{K_t} = \underset{I}{Max} \quad E_t\left[A_t - i_t + \beta\frac{\Lambda_{t+1}}{\Lambda_t}\frac{V_{t+1}}{K_{t+1}}g\left(i_t\right)\right] \tag{9.139}$$

Together, I can write

$$\frac{V_t}{K_t} = A_t - i_t + \frac{g\left(i_t\right)}{g'\left(i_t\right)} \tag{9.140}$$

Hence, the price dividend ratio is

$$\begin{aligned}
\frac{V_t - D_t}{D_t} &= \frac{V_t/K_t}{D_t/K_t} - 1 && (9.141)\\
&= \frac{A_t - i_t + \frac{g(i_t)}{g'(i_t)}}{A_t - i_t} - 1 && (9.142)\\
&= \frac{g\left(i_t\right)}{g'\left(i_t\right)\left(A_t - i_t\right)} && (9.143)
\end{aligned}$$

Further, the decomposition of the asset return can be derived,

$$\begin{aligned}
E_t\left(\frac{D_{t+1}}{V_t - D_t}\right) &= E_t\left(\frac{D_{t+1}/K_{t+1}}{V_t/K_t - D_t/K_t}\cdot g\left(i_t\right)\right) && (9.144)\\
&= E_t\left(\frac{A_{t+1} - i_{t+1}}{A_t - i_t + \frac{g(i_t)}{g'(i_t)} - A_t + i_t}\cdot g\left(i_t\right)\right) && (9.145)\\
&= g'\left(i_t\right)E_t\left(A_{t+1} - i_{t+1}\right) && (9.146)\\
& && (9.147)\\
E_t\left(\frac{V_{t+1} - D_{t+1}}{V_t - D_t}\right) &= E_t\left(\frac{V_{t+1}/K_{t+1} - D_{t+1}/K_{t+1}}{V_t/K_t - D_t/K_t}\cdot g\left(i_t\right)\right) && (9.148)\\
&= E_t\left(\frac{A_{t+1} - i_{t+1} + \frac{g(i_{t+1})}{g'(i_{t+1})} - A_{t+1} + i_{t+1}}{A_t - i_t + \frac{g(i_t)}{g'(i_t)} - A_t + i_t}\cdot g\left(i_t\right)\right) && (9.149)\\
&= g'\left(i_t\right)E_t\left(\frac{g\left(i_{t+1}\right)}{g'\left(i_{t+1}\right)}\right) && (9.150)
\end{aligned}$$

9.15 Roots of the Equation (6.10)

Again, it is necessary to develop proposition 10 to guarantee that the equation (6.7) has the unique root.

Proposition 10. *The following condition ensures that equation* (6.7) *has the unique BGP.*

$$\frac{Ag'(0)+g(0)}{g(0)^{\rho}} > \frac{1}{\beta} > g(A)^{1-\rho} \tag{9.151}$$

Proof. The left hand side of the equation (6.7) has the derivative with respect to i as

$$g'(i)\,g(i)^{\rho}\left[(A-i)\left(\frac{g''(i)}{g'(i)}-\rho\frac{g'(i)}{g(i)}\right)-\rho\right]<0 \tag{9.152}$$

The relation with 0 can be draw under the condition of $A-i>0$, $g(i)>0$, $g'(i)>0$ and $g''(i)<0$.

As a result, the left hand side of the equation (6.7) is monotonically decreasing with the investment to capital ration i in the range of $(0,\,A)$. Since it is a continuous function, we only need to specify that the starting and ending points are at opposite sides of $1/\beta$. Therefore, the inequality in the proposition follows.

Q.E.D. ■

9.16 Proposition 3

For growth rate of consumption and output, I have

$$\frac{Y_{t+1}}{Y_t} = \frac{AK_{t+1}}{AK_t} \tag{9.153}$$

$$= g(\bar{i}) \tag{9.154}$$

$$\frac{C_{t+1}}{C_t} = \frac{(A-\bar{i})\,K_{t+1}}{(A-\bar{i})\,K_t} = g(\bar{i}) \tag{9.155}$$

$$DF_{t+1} = \left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right]^{\theta}\left(\frac{1}{R_{t+1}}\right)^{1-\theta} \tag{9.156}$$

$$= \left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right]^{\theta}\left[\left(A_{t+1} - i_{t+1} + \frac{g\left(i_{t+1}\right)}{g'\left(i_{t+1}\right)}\right)g'\left(i_t\right)\right]^{\theta-1} \tag{9.157}$$

$$= \left[\beta\left(g\left(i_t\right)\right)^{-\rho}\right]^{\theta}\left[\left(A_t - i_t\right)g'\left(i_t\right) + g\left(i_t\right)\right]^{\theta-1} \tag{9.158}$$

$$1/DF_{t+1} = \left[\beta\left(g\left(i_t\right)\right)^{-\rho}\right]^{-\theta}\left[\left(A_t - i_t\right)g'\left(i_t\right) + g\left(i_t\right)\right]^{1-\theta} \tag{9.159}$$

$$= \frac{1}{\beta^{\theta}}\left(g\left(i\right)\right)^{\theta\rho}\left[\left(A - i\right)g'\left(i\right) + g\left(i\right)\right]^{1-\theta} \tag{9.160}$$

$$= \frac{1}{\beta^{\theta}}\left(g\left(i\right)\right)^{\theta\rho}\left[\frac{g\left(i\right)^{\rho}}{\beta}\right]^{1-\theta} \tag{9.161}$$

$$= \frac{1}{\beta^{\theta}}\left(g\left(i\right)\right)^{\theta\rho}\frac{g\left(i\right)^{\rho-\theta\rho}}{\beta^{1-\theta}} \tag{9.162}$$

$$= \frac{1}{\beta}g\left(i\right)^{\rho} \tag{9.163}$$

9.17 Transversality Condition of Baseline Model

Jointly, in the equilibrium, the growth rate of the firm's market capitalization has to be capped by the household's discounting behaviour. If I represent the Euler equation in the balanced growth path as

$$DF \times R = 1 \tag{9.164}$$

Then, in the firms problem, the transversality condition is

$$\lim_{\tau\to\infty} DF^{\tau}\frac{\partial D_{\tau}}{\partial K_{\tau+1}}K_{\tau+1} = 0 \tag{9.165}$$

$$\lim_{\tau\to\infty} DF^{\tau}R^{\tau}R^{-\tau}\frac{\partial D_{\tau}}{\partial K_{\tau+1}}K_{\tau+1} = 0 \tag{9.166}$$

$$\lim_{\tau\to\infty} R^{-\tau}g'\left(i_{\tau}\right)^{-1}K_{\tau+1} = 0 \tag{9.167}$$

$$\lim_{\tau\to\infty} R^{-\tau}g'\left(i_{\tau}\right)^{-1}K_0 g\left(\bar{i}\right)^{\tau+1} = 0 \tag{9.168}$$

$$\lim_{\tau\to\infty}\left[\left(A - \bar{i}\right)g'\left(\bar{i}\right) + g\left(\bar{i}\right)\right]^{-\tau}g'\left(i_{\tau}\right)^{-1}K_0 g\left(\bar{i}\right)^{\tau+1} = 0 \tag{9.169}$$

$$\lim_{\tau\to\infty}\left[\frac{g\left(\bar{i}\right)}{\left(A - \bar{i}\right)g'\left(\bar{i}\right) + g\left(\bar{i}\right)}\right]^{\tau}\frac{g\left(\bar{i}\right)}{g'\left(\bar{i}\right)}K_0 = 0 \tag{9.170}$$

I use the expression for R derived in proposition 6 in the equilibrium in equality (9.169). Since i is a constant and $(A - i)\, g'(i) > 0$ in the equilibrium, the condition is satisfied.

References

Antolin-Diaz, J., Drechsel, T. & Petrella, I. (2017), 'Tracking the slowdown in long-run gdp growth', *Review of Economics and Statistics*.

Azariadis, C. & Drazen, A. (1990), 'Threshold externalities in economic development', *The Quarterly Journal of Economics* pp. 501–526.

Azariadis, C. & Stachurski, J. (2005), Chapter 5 poverty traps, *in* Philippe Aghion and Steven N. Durlauf, ed., 'Handbook of Economic Growth', Vol. Volume 1, Part A, Elsevier, pp. 295–384.

Bansal, R. & Yaron, A. (2004), 'Risks for the long run: A potential resolution of asset pricing puzzles', *The Journal of finance* **59**(4), 1481–1509.

Benhabib, J. & Farmer, R. E. (1999), 'Indeterminacy and sunspots in macroeconomics', *Handbook of macroeconomics* **1**, 387–448.

Benigno, G. & Fornaro, L. (2016), 'Stagnation traps', *Working Papers*.

Blanchard, O., Lorenzoni, G. & L'Huillier, J.-P. (2017), Short-Run effects of lower productivity growth. a twist on the secular stagnation hypothesis.

Boldrin, M., Christiano, L. J. & Fisher, J. D. (2001), 'Habit persistence, asset returns, and the business cycle', *American Economic Review* pp. 149–166.

Campbell, J. Y. (2003), 'Consumption-based asset pricing', *Handbook of the Economics of Finance* **1**, 803–887.

Christiano, L. J. & Harrison, S. G. (1999), 'Chaos, sunspots and automatic stabilizers', *Journal of Monetary Economics* **44**(1), 3–31.

Cochrane, J. H. (1991), 'Production-based asset pricing and the link between stock returns and economic fluctuations', *The Journal of Finance* **46**(1), 209–237.

Cochrane, J. H. (2005), *Financial Markets and the Real Economy*, Now Publishers Inc.

Eberly, J. C. & Wang, N. (2009), Reallocating and pricing illiquid capital: Two productive trees, *in* 'AFA 2010 Atlanta Meetings Paper'.

Eggertsson, G. B., Mehrotra, N. R. & Robbins, J. A. (2017), A model of secular stagnation: Theory and quantitative evaluation.

Eichengreen, B. (2017), 'A two-handed approach to secular stagnation: Some thoughts based on 1930s experience', *Journal of Policy Modeling*.

Epstein, L. G. & Zin, S. E. (1989), 'Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework', *Econometrica: journal of the Econometric Society* **57**(4), 937–969.

Fatas, A. (2000), 'Endogenous growth and stochastic trends', *Journal of monetary economics* **45**, 107.

Favero, C. A., Gozluklu, A. E. & Yang, H. (2016), 'Demographics and the behavior of interest rates', *IMF Economic Review* **64**(4), 732–776.

Fernald, J. G. & Jones, C. I. (2014), 'The future of us economic growth', *The American economic review* **104**(5), 44–49.

Gabaix, X. (2012), 'Variable rare disasters: An exactly solved framework for ten puzzles in macro-finance', *The Quarterly journal of economics* **127**(2), 645–700.

Gordon, R. J. (2015), 'Secular stagnation: A Supply-Side view', *The American economic review* **105**(5), 54–59.

Gourio, F. (2012), 'Disaster risk and business cycles', *The American economic review* **102**(6), 2734–2766.

Gruber, J. W. & Kamin, S. B. (2016), 'The corporate saving glut and falloff of investment spending in OECD economies', *IMF Economic Review* **64**(4), 777–799.

Hall, R. E. (1988), 'Intertemporal substitution in consumption', *The journal of political economy*.

Hamilton, J. D. (1989), 'A new approach to the economic analysis of nonstationary time series and the business cycle', *Econometrica: journal of the Econometric Society* **57**(2), 357–384.

Hayashi, F. (1982), 'Tobin's marginal q and average q: A neoclassical interpretation', *Econometrica* **50**(1), 213–224.

Jermann, U. J. (1998), 'Asset pricing in production economies', *Journal of monetary Economics* **41**(2), 257–275.

Jinnai, R. (2015), 'Innovation, product cycle, and asset prices', *Review of Economic Dynamics* **18**(3), 484–504.

Kogan, L. (2001), 'An equilibrium model of irreversible investment', *Journal of Financial Economics* **62**(2), 201–245.

Kogan, L. & Papanikolaou, D. (2012), 'Economic activity of firms and asset prices', *Annu. Rev. Financ. Econ.* **4**(1), 361–384.

Lucas, Jr., R. E. (1978), 'Asset prices in an exchange economy', *Econometrica: journal of the Econometric Society* **46**(6), 1429–1445.

Matsuyama, K. (1997), 'Complementarity, instability and multiplicity', *Japanese Economic Review* **48**(3), 240–265.

Mehra, R. & Prescott, E. C. (1985), 'The equity premium: A puzzle', *Journal of monetary economics* **15**(2), 145–161.

Pescatori, A. & Turunen, J. (2016), 'Lower for longer: Neutral rate in the US', *IMF Economic Review* **64**(4), 708–731.

Romer, P. M. (1986), 'Increasing returns and Long-Run growth', *The journal of political economy* **94**(5), 1002–1037.

Sajedi, R. & Thwaites, G. (2016), 'Why are real interest rates so low? the role of the relative price of investment goods', *IMF Economic Review* **64**(4), 635–659.

Summers, L. H. (2015), 'Demand side secular stagnation', *The American economic review* **105**(5), 60–65.

van Binsbergen, J. H., Fernández-Villaverde, J., Koijen, R. S. J. & Rubio-Ramírez, J. (2012), 'The term structure of interest rates in a DSGE model with recursive preferences', *Journal of monetary economics* **59**(7), 634–648.

Weil, P. (1989), 'The equity premium puzzle and the risk-free rate puzzle', *Journal of monetary economics* **24**(3), 401–421.

Woodford, M. (1986), ‘Stationary sunspot equilibria-the case of small fluctuations around a deterministic steady state’.