Optimal Portfolio Selection and Consumption Strategies under Stochastic Volatility

Lei GE[1] and Qiang ZHANG[2]

[1] *School of Finance, Southwestern University of Finance and Economics, Chengdu, China*
[2] *Department of Mathematics, City University of Hong Kong, Hong Kong*
Corresponding author: Lei Ge, leige365@outlook.com

Abstract. How to optimally allocate wealth on different financial instruments and how to determine the best consumption strategy are fundamental and practical issues for modern portfolio management. They are also very challenging tasks mathematically. Most of the studies on this topic are for the settings in which the volatilities of the risky assets are constant. However, empirical studies show that the volatilities are stochastic. This increases the mathematical difficulties dramatically. We study this important and difficult topic in a general and practical setting: stochastic volatility, incomplete markets, finite investment horizons, and constant relative risk aversion (CRRA) utility. There is no analytical solution under this setting, and even no numerical solutions are available in the literature. Based on the method of dynamic programming, the solution to this difficult problem is governed by a nonlinear and inhomogeneous partial differential equation, namely, the Hamilton-Jacobi-Bellman equation. In this paper, we present an accurate and efficient numerical algorithm for solving this dynamic optimization problem.

Keywords: portfolio management, consumption strategy, stochastic volatility, optimal investment, stochastic control, Hamilton-Jacobi-Bellman equation

Introduction

How to optimally allocate assets and optimally consume are extremely important and difficult topics in portfolio management [1, 17, 10]. These topics are important not only for theoretical consideration, but also for applications in financial industry. Empirical studies have shown that volatilities of risky assets should be modeled as stochastic rather than deterministic. This adds further complication to the problem. The optimal asset allocation and optimal consumption strategies are governed by the Hamilton-Jacobi-Bellman (HJB) equation. Due to the nonlinearity and inhomogeneity of this partial differential equation, no exact solution has been found. Furthermore, even numerical solutions are not available. In this paper, we present an accurate and efficient numerical method for solving this equation, and generate the first set of accurate numerical solutions for this problem.

Due to the importance of portfolio selection under stochastic volatilities, several important theoretical works have been carried out and exact solutions have been obtained under certain special settings, such as no consumption ([11, 19, 13]), complete markets which means that the stock movement and the volatility movement are either perfectly correlated or perfectly anti-correlated ([19, 13, 21, 18]), or when investors have unit elasticity of intertemporal substitution of consumption [3].

In this paper, we consider this optimal stochastic control problem under a general setting: stochastic volatility, incomplete markets, finite investment horizons and CRRA utility. Our numerical method combines a three-level Crank-Nicolson scheme and Richardson's extrapolation technique. The Crank-Nicolson scheme has second-order accuracy in terms of discretization error and Richardson's extrapolation technique further improves the accuracy. We verify that our numerical method is accurate and efficient.

This paper is organized as follows. In Section 1, we describe the model for financial market, the stochastic control optimization procedure and the governing HJB equation for the optimal asset allocation and consumption strategies. In Section 2, we present our numerical method for solving the HJB equation. In Section 3, we verify the accuracy and the efficiency of our numerical method and present accurate numerical solutions for the optimal asset allocation strategy and the optimal consumption strategy. In the last section, we present our conclusions.

1. Financial market and stochastic control

We consider a market consisting of one riskless asset B_t, whose price is governed by

$$dB_t = rB_t dt,$$

with a constant risk-free interest rate r and a risky asset S_t modeled as

$$dS_t = S_t\big[\mu(v_t) + \sigma(v_t)dW_t^S\big]. \tag{1.1}$$

In (1.1), $\mu(v_t)$ and $\sigma(v_t)$ are the return and the stochastic volatility of the stock price S_t, respectively. $v_t = \sigma_t^2$ is the stochastic variance of S_t. Empirical studies show presence of mean reversion in the stock movements [16]. Heston model [9] is selected for v_t, namely,

$$dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t}dW_t^v. \tag{1.2}$$

Here dW_t^v and dW_t^S are the increments of the Wiener processes under a probability P. The correlation between dW_t^v and dW_t^S is ρ, namely $\text{Corr}(dv_t/v_t, dS_t/S_t) = \rho dt$. We assume ρ is a constant. In (1.2), θ is the long-run average variance (i.e., as t tends to infinity, the expected value of v_t tends to θ), κ is the rate at which v_t reverts to θ, ξ is the volatility of the stock variance v_t. The parameters κ, θ, ξ are positive constants and need to satisfy the Feller condition, $2\kappa\theta > \xi^2$, to ensure that v_t is strictly positive. The risk premia is defined as

$$A = \frac{\mu_t - r}{\sigma_t^2} \equiv \frac{\mu_t - r}{v_t}.$$

Following [14], [9], [1], [6] and [12], we assume A is a constant. This means the stock excess return is proportional to the stock variance.

Consider an investor who has an initial wealth w_0 and needs to determine strategies for asset allocation and consumption over an investment horizon $[0,T]$. Let w_t be the investor's wealth at time t. The strategies are consist of an asset allocation rate ϕ_t and a consumption rate c_t, which means he/she allocates $\phi_t w_t$ to the risky asset and $(1-\phi_t)w_t$ to the riskless asset at time t and consumes $c_t dt$ over the time interval $[t, t+dt]$. Thus, under the strategies ϕ_t and c_t, the wealth process is governed by

$$dw_t = \frac{\phi_t w_t}{S_t}dS_t + (1-\phi_t)w_t r dt - c_t dt.$$

The goal is to maximize the expected utilities over the investment horizon, namely,

$$\sup_{\phi_t, c_t} E\left[\int_0^T \alpha e^{-\beta t}u_1(c_t)dt + (1-\alpha)e^{-\beta T}u_2(w_T)\right]. \tag{1.3}$$

In (1.3), ϕ_t and c_t are control variables for this optimization problem. E is the expectation operator under the probability P. β is the subjective discount rate, namely, the time preference of the investor. The larger β is, the more weight the investor puts on the present than on the future. The parameter α determines the relative importance between the intertemporal consumption and the terminal wealth. $u_1(\cdot)$ and $u_2(\cdot)$ are the investor's utility functions which measure the investor's degree of satisfaction with the outcomes from intertemporal consumption and terminal wealth, respectively.

CRRA utility functions have been widely adopted for modeling investors' behavior. Therefore, we adopt the CRRA utility function for $u_1(\cdot)$ and $u_2(\cdot)$:

$$u_1(c_t) = \begin{cases} \dfrac{a_c c_t^{1-\gamma}}{1-\gamma}, & \text{for } \gamma \neq 1, \\ a_c \log(c_t), & \text{for } \gamma = 1 \end{cases}$$

and

$$u_2(w_T) = \begin{cases} \dfrac{a_w w_T^{1-\gamma}}{1-\gamma}, & \text{for } \gamma \neq 1, \\ a_w \log(w_T), & \text{for } \gamma = 1, \end{cases}$$

where γ, a_c and a_w are positive constants. Since $u_1(\cdot)$ and $u_2(\cdot)$ stand for the intertemporal consumption utility and the terminal wealth utility of the same investor, we use the same γ in $u_1(\cdot)$ and $u_2(\cdot)$. However, a_c and a_w can be different, since c and w have different dimensions.

Let $V(t, w, v)$ be the value function with a terminal condition

$$V(T, w_T, v_T) = (1-\alpha)e^{-\beta T}u_2(w_T).$$

Based on the HJB dynamic programming procedure, V is governed by

$$\begin{aligned} 0 = \sup_{\phi, c} \Big[& \alpha e^{-\beta t}u_1(c) + V_t + \big(rw + \phi w A v - c\big)V_w + \kappa(\theta - v)V_v \\ & + \frac{1}{2}\phi^2 w^2 v V_{ww} + \phi w \rho \xi v V_{wv} + \frac{1}{2}\xi^2 v V_{vv}\Big], \end{aligned} \tag{1.4}$$

with the optimal strategies ϕ^* and c^* determined by

$$\phi^* = -\frac{AV_w + \rho\xi V_{wv}}{wV_{ww}}, \tag{1.5}$$

$$c^* = \left(\frac{V_w}{a_c \alpha e^{-\beta t}}\right)^{-\frac{1}{\gamma_c}}. \tag{1.6}$$

After substituting the expressions (1.5) and (1.6) into (1.4), one obtains an equation for the value function V

$$\frac{\gamma}{1-\gamma}\big(a_c\alpha e^{-\beta t}\big)^{\frac{1}{\gamma}} V_w^{\frac{1-\gamma}{-\gamma}} + rwV_w + V_t + \kappa(\theta - v)V_v + \frac{1}{2}\xi^2 v V_{vv} - \frac{1}{2}v\frac{\big(AV_w + \rho\xi V_{wv}\big)^2}{V_{ww}} = 0. \tag{1.7}$$

Choosing a functional form

$$V(\tau, w, v) = e^{-\beta(T-\tau)}\frac{a_w w^{1-\gamma}}{1-\gamma}f(\tau, v)^{\gamma},$$

where $\tau = T - t$, Eq. (1.7) becomes

$$\begin{aligned} 0 = & - f_\tau + \frac{1}{2}\xi^2 v f_{vv} + \left(\kappa(\theta - v) + \frac{1-\gamma}{\gamma}A\rho\xi v\right) f_v - \frac{1}{2}(1-\gamma)(1-\rho^2)\xi^2 v\frac{f_v^2}{f} \\ & + \left(\frac{(1-\gamma)A^2}{2\gamma^2}v + \frac{(1-\gamma)r}{\gamma} - \frac{\beta}{\gamma}\right) f + \left(\frac{\alpha a_c}{a_w}\right)^{\frac{1}{\gamma}} \end{aligned} \tag{1.8}$$

with

$$f(0, v) = (1-\alpha)^{\frac{1}{\gamma}}, \tag{1.9}$$

and Eqs. (1.5) and (1.6) become

$$\phi^* = \frac{A}{\gamma} + \rho\xi\frac{f_v}{f} \tag{1.10}$$

$$c^*/w = \left(\frac{\alpha a_c}{a_w}\right)^{\frac{1}{\gamma}} f^{-1}. \tag{1.11}$$

Equation (1.8) is a nonlinear and inhomogeneous partial differential equation. Since no closed-form solution is available for this equation, numerical computation plays a critical role for studying this important practical problem in modern finance. However, there are even no numerical solutions available in the literature.

2. Numerical method

In this section, we develop a numerical method for solving Eq. (1.8). For the sake of conciseness of our expressions, we rewrite Eq. (1.8) as

$$-f_\tau + a_1 v f_{vv} + (a_2 v + a_3) f_v + a_4 v \frac{f_v^2}{f} + (a_5 v + a_6) f + \alpha^{\frac{1}{\gamma}} a_7 = 0 \tag{2.1}$$

with the initial condition $f(0,v) = (1-\alpha)^{\frac{1}{\gamma}}$, where

$$a_1 = \frac{1}{2}\xi^2, \qquad a_2 = \frac{1-\gamma}{\gamma} A\rho\xi - \kappa, \qquad a_3 = \kappa\theta, \qquad a_4 = -\frac{1}{2}(1-\gamma)\xi^2(1-\rho^2),$$
$$a_5 = \frac{1-\gamma}{2\gamma^2} A^2, \qquad a_6 = \frac{(1-\gamma)r}{\gamma} - \frac{\beta}{\gamma}, \qquad a_7 = \left(\frac{a_c}{a_w}\right)^{\frac{1}{\gamma}}.$$

2.1. Crank-Nicolson scheme and Richardson's extrapolation

We use a three-level Crank-Nicolson scheme (see [8], [20], [7] and [2]) of second-order accuracy to solve the nonlinear and inhomogeneous partial differential equation given by Eq. (2.1), and use Richardson's extrapolation technique for further improving accuracy. Numerically, one can only solve Eq. (2.1) over a finite domain $v \in [0, v_{max}]$. Since the boundary conditions at $v = 0$ and at $v = v_{max}$ are not known, we use one-sided difference method at these two numerical boundaries. Step sizes $\Delta\tau$ and Δv are used to discretize τ and v, respectively. Thus $\tau = n\Delta\tau$ and $v = m\Delta v$. We adopt the standard notation $f_m^n = f(n\Delta\tau, m\Delta v)$.

The three-level Crank-Nicolson scheme involves the levels $n-1$, n and $n+1$. It is straightforward to discretize all linear terms in Eq. (2.1) with second-order errors, namely,

$$\begin{aligned}
(f_\tau)_m^n &= \frac{f_m^{n+1} - f_m^{n-1}}{2\Delta\tau} + O(\Delta\tau^2),\\
(f_v)_m^n &= \frac{1}{2}\left(\frac{f_{m+1}^{n+1} - f_{m-1}^{n+1}}{2\Delta v} + \frac{f_{m+1}^{n-1} - f_{m-1}^{n-1}}{2\Delta v}\right) + O(\Delta\tau^2) + O(\Delta v^2),\\
(f_{vv})_m^n &= \frac{1}{2}\left(\frac{f_{m+1}^{n+1} - 2f_m^{n+1} + f_{m-1}^{n+1}}{\Delta v^2} + \frac{f_{m+1}^{n-1} - 2f_m^{n-1} + f_{m-1}^{n-1}}{\Delta v^2}\right) + O(\Delta\tau^2) + O(\Delta v^2).
\end{aligned} \tag{2.2}$$

The nonlinear term $\frac{{f_v}^2}{f}$ has two factors $\frac{f_v}{f}$ and f_v. We discretize the factor $\frac{f_v}{f}$ at level n and approximate the factor f_v as an average between f_v at level $n-1$ and that at level $n+1$, namely,

$$\begin{aligned}
\left(\frac{{f_v}^2}{f}\right)_m^n &= \left(\frac{f_v}{f}\right)_m^n \frac{1}{2}\left((f_v)_m^{n+1} + (f_v)_m^{n-1}\right) + O(\Delta\tau^2)\\
&= \frac{f_{m+1}^n - f_{m-1}^n}{4\Delta v f_m^n}\left(\frac{f_{m+1}^{n+1} - f_{m-1}^{n+1}}{2\Delta v} + \frac{f_{m+1}^{n-1} - f_{m-1}^{n-1}}{2\Delta v}\right) + O(\Delta\tau^2) + O(\Delta v^2).
\end{aligned} \tag{2.3}$$

This discretization scheme leads to a set of linear equations. Based on the expressions given by Eqs. (2.2) and (2.3), Eq. (2.1) can be discretized as

$$e_1(m) f_{m-1}^{n+1} + e_2(m) f_m^{n+1} + e_3(m) f_{m+1}^{n+1} = e_4(m) + O(\Delta\tau^2) + O(\Delta v^2), \tag{2.4}$$

for $0 < m < M$, $n \geq 2$, where M is the maximal value of m and

$$\begin{aligned}
e_1(m) &= \frac{a_1 m}{\Delta v} - \frac{\lambda(m)}{2\Delta v}, \quad e_2(m) = \frac{-2a_1 m}{\Delta v} - \frac{1}{\Delta\tau} + a_5 m\Delta v + a_6, \quad e_3(m) = \frac{a_1 m}{\Delta v} + \frac{\lambda(m)}{2\Delta v},\\
e_4(m) &= -\frac{1}{\Delta\tau} f_m^{n-1} - 2\alpha^{\frac{1}{\gamma}} a_7 - \frac{a_1 m}{\Delta v}(f_{m+1}^{n-1} - 2f_m^{n-1} + f_{m-1}^{n-1}) - \frac{\lambda(m)}{2\Delta v}(f_{m+1}^{n-1} - f_{m-1}^{n-1}) - (a_5 m\Delta v + a_6) f_m^{n-1}
\end{aligned}$$

with

$$\lambda(m) = a_2 m\Delta v + a_3 + \frac{a_4 m(f_{m+1}^n - f_{m-1}^n)}{2f_m^n}.$$

Since Eq. (2.4) is not applicable to the boundaries at $m = 0$ and $m = M$, we use one-sided difference approximation at these two boundaries. It is straightforward to show that, at $m = 0$, we have

$$
\begin{aligned}
f_v(0,\tau) &= \frac{1}{4\Delta v}\big(-3f_0^{n+1} + 4f_1^{n+1} - f_2^{n+1} - 3f_0^{n-1} + 4f_1^{n-1} - f_2^{n-1}\big) + O(\Delta\tau^2) + O(\Delta v^2),\\
f_{vv}(0,\tau) &= \frac{1}{2\Delta v^2}\big(2f_0^{n+1} - 5f_1^{n+1} + 4f_2^{n+1} - f_3^{n+1} + 2f_0^{n-1} - 5f_1^{n-1} + 4f_2^{n-1} - f_3^{n-1}\big) + O(\Delta\tau^2) + O(\Delta v^2),\\
\frac{f_v^2}{f}(0,\tau) &= \frac{-3f_0^n + 4f_1^n - f_2^n}{8\Delta v^2 f_0^n}\big(-3f_0^{n+1} + 4f_1^{n+1} - f_2^{n+1} - 3f_0^{n-1} + 4f_1^{n-1} - f_2^{n-1}\big) + O(\Delta\tau^2) + O(\Delta v^2)
\end{aligned}
$$

and, at $m = M$, we have

$$
\begin{aligned}
f_v(v_{max},\tau) =& \frac{1}{4\Delta v}\big(3f_M^{n+1} - 4f_{M-1}^{n+1} + f_{M-2}^{n+1} + 3f_M^{n-1} - 4f_{M-1}^{n-1} + f_{M-2}^{n-1}\big) + O(\Delta\tau^2) + O(\Delta v^2),\\
f_{vv}(v_{max},\tau) =& \frac{1}{2\Delta v^2}\big(2f_M^{n+1} - 5f_{M-1}^{n+1} + 4f_{M-2}^{n+1} - f_{M-3}^{n+1} + 2f_M^{n-1} - 5f_{M-1}^{n-1} + 4f_{M-2}^{n-1} - f_{M-3}^{n-1}\big)\\
&+ O(\Delta\tau^2) + O(\Delta v^2),\\
\frac{f_v^2}{f}(v_{max},\tau) =& \frac{3f_M^n - 4f_{M-1}^n + f_{M-2}^n}{8\Delta v^2 f_M^n}\big(3f_M^{n+1} - 4f_{M-1}^{n+1} + f_{M-2}^{n+1} + 3f_M^{n-1} - 4f_{M-1}^{n-1} + f_{M-2}^{n-1}\big) + O(\Delta\tau^2) + O(\Delta v^2).
\end{aligned}
$$

By substituting these expressions into Eq. (2.1), we have

$$
d_1(0)f_0^{n+1} + d_2(0)f_1^{n+1} + d_3(0)f_2^{n+1} + d_4(0)f_3^{n+1} = d_5(0) + O(\Delta\tau^2) + O(\Delta v^2) \tag{2.5}
$$

and

$$
d_4(M)f_{M-3}^{n+1} + d_3(M)f_{M-2}^{n+1} + d_2(M)f_{M-1}^{n+1} + d_1(M)f_M^{n+1} = d_5(M) + O(\Delta\tau^2) + O(\Delta v^2), \tag{2.6}
$$

where

$$
\begin{aligned}
d_1(0) &= -\frac{3\lambda(0)}{2\Delta v} - \frac{1}{\Delta\tau} + a_6, & d_2(0) &= \frac{2\lambda(0)}{\Delta v},\\
d_3(0) &= -\frac{\lambda(0)}{2\Delta v}, & d_4(0) &= 0,\\
d_5(0) &= -\frac{1}{\Delta\tau}f_0^{n-1} - 2\alpha^{\frac{1}{\gamma}}a_7 - \frac{\lambda(0)}{2\Delta v}(-3f_0^{n-1} + 4f_1^{n-1} - f_2^{n-1}) - a_6 f_0^{n-1}
\end{aligned}
$$

and

$$
\begin{aligned}
d_1(M) =& \frac{2a_1M}{\Delta v} + \frac{3\lambda(M)}{2\Delta v} - \frac{1}{\Delta\tau} + a_5M\Delta v + a_6, & d_2(M) &= \frac{-5a_1M}{\Delta v} - \frac{2\lambda(M)}{\Delta v},\\
d_3(M) =& \frac{4a_1M}{\Delta v} + \frac{\lambda(M)}{2\Delta v}, & d_4(M) &= \frac{-a_1M}{\Delta v},\\
d_5(M) =& -\frac{1}{\Delta\tau}f_M^{n-1} - 2\alpha^{\frac{1}{\gamma}}a_7 - \frac{a_1M}{\Delta v}(2f_M^{n-1} - 5f_{M-1}^{n-1} + 4f_{M-2}^{n-1} - f_{M-3}^{n-1})\\
&- \frac{\lambda(M)}{2\Delta v}(3f_M^{n-1} - 4f_{M-1}^{n-1} + f_{M-2}^{n-1}) - (a_5M\Delta v + a_6)f_M^{n-1}
\end{aligned}
$$

with

$$
\lambda(0) = a_3 \quad\text{and}\quad \lambda(M) = a_2M\Delta v + a_3 + \frac{a_4M(3f_M^n - 4f_{M-1}^n + f_{M-2}^n)}{2f_M^n}.
$$

From Eqs. (2.4), (2.5) and (2.6), the numerical solution of Eq. (2.1) for $n \geq 2$ is determined by the following system of linear equations:

$$
\begin{bmatrix}
d_1(0) & d_2(0) & d_3(0) & d_4(0) & & & \\
e_1(1) & e_2(1) & e_3(1) & & & & \\
 & e_1(2) & e_2(2) & e_3(2) & & & \\
 & & \ddots & \ddots & \ddots & & \\
 & & & e_1(M-1) & e_2(M-1) & e_3(M-1) \\
 & & d_4(M) & d_3(M) & d_2(M) & d_1(M)
\end{bmatrix}
\begin{bmatrix} f_0^{n+1} \\ f_1^{n+1} \\ f_2^{n+1} \\ \vdots \\ f_{M-1}^{n+1} \\ f_M^{n+1} \end{bmatrix}
=
\begin{bmatrix} d_5(0) \\ e_4(1) \\ e_4(2) \\ \vdots \\ e_4(M-1) \\ d_5(M) \end{bmatrix}. \tag{2.7}
$$

After eliminating $d_3(0)$, $d_4(0)$, $d3(M)$ and $d_4(M)$, Eq. (2.7) can be transformed into the following tri-diagonal matrix form for $n \geq 2$:

$$\begin{bmatrix} e_2(0) & e_3(0) & & & & \\ e_1(1) & e_2(1) & e_3(1) & & & \\ & e_1(2) & e_2(2) & e_3(2) & & \\ & & \ddots & \ddots & \ddots & \\ & & & e_1(M-1) & e_2(M-1) & e_3(M-1) \\ & & & & e_1(M) & e_2(M) \end{bmatrix} \begin{bmatrix} f_0^{n+1} \\ f_1^{n+1} \\ f_2^{n+1} \\ \vdots \\ f_{M-1}^{n+1} \\ f_M^{n+1} \end{bmatrix} = \begin{bmatrix} e_4(0) \\ e_4(1) \\ e_4(2) \\ \vdots \\ e_4(M-1) \\ e_4(M) \end{bmatrix}, \tag{2.8}$$

where

$$\begin{aligned}
e_2(0) =& d_1(0) - \frac{e_3(2)d_3(0) - e_2(2)d_4(0)}{e_3(2)e_3(1)} e_1(1), \\
e_3(0) =& d_2(0) - \frac{d_4(0)}{e_3(2)} e_1(2) - \frac{e_3(2)d_3(0) - e_2(2)d_4(0)}{e_3(2)e_3(1)} e_2(1), \\
e_4(0) =& d_5(0) - \frac{d_4(0)}{e_3(2)} e_4(2) - \frac{e_3(2)d_3(0) - e_2(2)d_4(0)}{e_3(2)e_3(1)} e_4(1), \\
e_1(M) =& d_2(M) - \frac{d_4(M)}{e_1(M-2)} e_3(M-2) - \frac{e_1(M-2)d_3(M) - e_2(M-2)d_4(M)}{e_1(M-2)e_1(M-1)} e_2(M-1), \\
e_2(M) =& d_1(M) - \frac{e_1(M-2)d_3(M) - e_2(M-2)d_4(M)}{e_1(M-2)e_1(M-1)} e_3(M-1), \\
e_4(M) =& d_5(M) - \frac{d_4(M)}{e_1(M-2)} e_4(M-2) - \frac{e_1(M-2)d_3(M) - e_2(M-2)d_4(M)}{e_1(M-2)e_1(M-1)} e_4(M-1).
\end{aligned}$$

The method given by Eqs. (2.2) and (2.3) is a two-step method, namely, f^{n+1} depends on f^n and f^{n-1}. At the zeroth step, f^0 is given by the initial condition Eq. (1.9), namely, $f_m^0 = (1-\alpha)^{\frac{1}{\gamma}}$ for $0 \leq m \leq M$. We now determine f^1, the solution at the first step. Performing a Taylor expansion on $f(\Delta\tau, v)$ at $\tau = 0$ gives

$$f(\Delta\tau, v) = f(0, v) + f_\tau(0, v)\Delta\tau + \frac{1}{2} f_{\tau\tau}(0, v)\Delta\tau^2 + \frac{1}{6} f_{\tau\tau\tau}(0, v)\Delta\tau^3 + O(\Delta\tau^4), \tag{2.9}$$

where $f(0, v)$ is given by Eq. (1.9) and $f_\tau(0, v)$, $f_{\tau\tau}(0, v)$ and $f_{\tau\tau\tau}(0, v)$ can be determined analytically from Eq. (2.1):

$$\begin{aligned}
f_\tau(0, v) =& (a_5 v + a_6) f(0, v) + \alpha^{\frac{1}{\gamma}} a_7, \\
f_{\tau\tau}(0, v) =& \left(a_5^2 v^2 + (a_2 a_5 + 2a_5 a_6) v + a_3 a_5 + a_6^2\right) f(0, v) + (a_5 v + a_6)\alpha^{\frac{1}{\gamma}} a_7, \\
f_{\tau\tau\tau}(0, v) =& \Big[a_5^3 v^3 + \left(3a_2 a_5^2 + 3a_6 a_5^2\right) v^2 + \left(a_5 a_2^2 + 3a_5 a_6 a_2 + 2a_1 a_5^2 + 3a_3 a_5^2 + 2a_4 a_5^2 + 3a_5 a_6^2\right) v \\
& + a_2 a_3 a_5 + 3a_3 a_6 a_5 + a_6^3\Big] f(0, v) + (a_2 v + a_3) a_5 \alpha^{\frac{1}{\gamma}} a_7 + (a_5 v + a_6)^2 \alpha^{\frac{1}{\gamma}} a_7.
\end{aligned}$$

The details of derivations for $f_\tau(0, v)$, $f_{\tau\tau}(0, v)$ and $f_{\tau\tau\tau}(0, v)$ are given in Appendix A. From Eq. (2.9), f_m^1 is given by

$$f_m^1 = f_m^0 + f_\tau(0, m\Delta v)\Delta\tau + \frac{1}{2} f_{\tau\tau}(0, m\Delta v)\Delta\tau^2 + \frac{1}{6} f_{\tau\tau\tau}(0, m\Delta v)\Delta\tau^3 \tag{2.10}$$

with an error of $O(\Delta\tau^4)$.

Knowing f, the numerical solutions of optimal portfolio and consumption rules can be obtained from Eqs. (1.10) and (1.11):

$$\phi_m^{n\,*} = \frac{A}{\gamma} + \rho\xi \frac{(f_v)_m^n}{f_m^n}, \tag{2.11}$$

$$c_m^{n\,*}/w = \left(\frac{\alpha a_c}{a_w}\right)^{\frac{1}{\gamma}} \left(f_m^n\right)^{-1}, \tag{2.12}$$

for $0 \le m \le M$, where f_m^n is given by Eq. (2.8) and $(f_v)_m^n$ is given by

$$(f_v)_m^n = \begin{cases} \dfrac{-3f_0^n + 4f_1^n - f_2^n}{2\Delta v}, & \text{for } m = 0, \\ \dfrac{f_{m+1}^n - f_{m-1}^n}{2\Delta v}, & \text{for } 0 < m < M, \\ \dfrac{3f_M^n - 4f_{M-1}^n + f_{M-2}^n}{2\Delta v}, & \text{for } m = M. \end{cases}$$

In summary, our numerical solutions for f_m^0 and f_m^1 are determined by Eqs. (1.9) and (2.10), respectively, and the numerical solutions for f_m^n with $n \ge 2$ are determined by Eq. (2.8). The numerical solution of f obtained by the three-level Crank-Nicolson scheme has an accuracy of $O(\Delta\tau^2) + O(\Delta v^2)$.

2.2. Performing Richardson's extrapolation

To further improve the accuracy of the numerical method, we apply Richardson's extrapolation technique to f. We will choose Δv proportional to $\Delta\tau$. Let $f(\tau_n, v_m, \Delta\tau)$ represent f_m^n obtained by Eq. (2.8) with a step size $\Delta\tau$. Then

$$f(\tau_n, v_m, \Delta\tau) = f_{exact}(\tau, v) + C_1\Delta\tau^2 + C_2\Delta\tau^3 + O(\Delta\tau^4),$$

where f_{exact} is the exact value. We perform two computations with the step sizes $\Delta\tau$ and $\frac{\Delta\tau}{2}$, respectively. Then we have the following two equations:

$$f(\tau_n, v_m, \Delta\tau) = f_{exact}(\tau, v) + C_1\Delta\tau^2 + C_2\Delta\tau^3 + O(\Delta\tau^4), \tag{2.13}$$

$$f(\tau_{2n}, v_{2m}, \frac{\Delta\tau}{2}) = f_{exact}(\tau, v) + \frac{C_1}{2^2}(\Delta\tau^2) + \frac{C_2}{2^3}\Delta\tau^3 + O(\Delta\tau^4). \tag{2.14}$$

From Eqs. (2.13) and (2.14), we solve f_{exact} and obtain an expression based on Richardson's extrapolation technique:

$$f_{extpl.}(n\Delta\tau, m\Delta v) = \frac{4}{3}f(\tau_{2n}, v_{2m}, \frac{\Delta\tau}{2}) - \frac{1}{3}f(\tau_n, v_m, \Delta\tau) = f_{exact} + O(\Delta\tau^3). \tag{2.15}$$

After substituting $f_{extpl.}$ into Eqs. (2.11) and (2.12), we obtain the expressions for ϕ^* and c^*/w with an accuracy of $O(\Delta\tau^3)$:

$$\phi^*{}_{extpl.} = \frac{A}{\gamma} + \rho\xi\frac{(f_{extpl.})_v}{f_{extpl.}}, \tag{2.16}$$

$$c^*{}_{extpl.}/w = \left(\frac{\alpha a_c}{a_w}\right)^{\frac{1}{\gamma}}\left(f_{extpl.}\right)^{-1}, \tag{2.17}$$

where $f_{extpl.}$ is given by Eq. (2.15), and $(f_{extpl.})_v$ is given by

$$\Big((f_{extpl.})_v\Big)_m^n = \begin{cases} \dfrac{1}{6\Delta v}\left[-9(f_{extpl.})_m^n + 16(f_{extpl.})_{m+1}^n - 8(f_{extpl.})_{m+2}^n + (f_{extpl.})_{m+4}^n\right], & \text{for } m \le 1, \\ \dfrac{1}{6\Delta v}\left[(f_{extpl.})_{m-2}^n - 4(f_{extpl.})_{m-1}^n + 4(f_{extpl.})_{m+1}^n - (f_{extpl.})_{m+2}^n\right], & \text{for } 1 < m < M-1, \\ \dfrac{1}{6\Delta v}\left[9(f_{extpl.})_m^n - 16(f_{extpl.})_{m-1}^n + 8(f_{extpl.})_{m-2}^n - (f_{extpl.})_{m-4}^n\right], & \text{for } m \ge M-1. \end{cases}$$

We summarize the procedure for obtaining the numerical solutions of f, ϕ^*, c^*, $f_{extpl.}$, $\phi^*{}_{extpl.}$ and $c^*{}_{extpl.}$. Here we choose $\Delta v = \Delta\tau$.

Step 1: Initialize f by the initial condition Eq. (1.9), namely, $f_m^0 = (1-\alpha)^{\frac{1}{\gamma}}$ for $0 \le m \le M$.
Step 2: Initialize f_m^1 by Eq. (2.10) for $0 \le m \le M$.
Step 3: For $n > 2$: knowing f_m^{n-1} and f_m^n, for $0 \le m \le M$, f_m^{n+1} can be determined from Eq. (2.8), which is in a tri-diagonal form and can be easily and efficiently solved. f_m^n has an accuracy of $O(\Delta\tau^2)$.
Step 4: To obtain ϕ^* and c^*, we substitute f_m^n from Steps 1–3 into Eqs. (2.11) and (2.12). This provides the numerical solutions for optimal strategies ϕ^{*n}_m and c^{*n}_m without extrapolation, which have accuracy of $O(\Delta\tau^2)$.

Step 5: To obtain $f_{extpl.}$, ${\phi^*}_{extpl.}$ and ${c^*}_{extpl.}$, we repeat Steps 1–3 with the step size $\frac{\Delta\tau}{2}$ to obtain $f(\tau_{2n}, v_{2m}, \frac{\Delta\tau}{2})$. Then, from Eqs. (2.15), (2.16) and (2.17), we obtain $f_{extpl.}$, ${\phi^*}_{extpl.}$ and ${c^*}_{extpl.}$, all of which have accuracy of $O(\Delta\tau^3)$.

In the next section, we will verify the accuracy of the numerical solutions without extrapolation and those with extrapolation.

3. Validation study of numerical method

Equation (2.1) is an inhomogeneous equation. However, since the inhomogeneous term $\alpha^{\frac{1}{\gamma}}a_7$ affects neither the stability nor the accuracy of the three-level Crank-Nicolson method, it is sufficient to conduct validation studies for the corresponding homogeneous equation, namely, for the case of $\alpha = 0$. Let $\hat{f}$ be the solution of the homogeneous equation of Eq. (2.1), namely, the case of $\alpha = 0$. Then $\hat{f}$ satisfies

$$-(\hat{f})_\tau + a_1 v(\hat{f})_{vv} + (a_2 v + a_3)(\hat{f})_v + a_4 v\frac{(\hat{f})_v^2}{\hat{f}} + (a_5 v + a_6)\hat{f} = 0 \tag{3.1}$$

with the initial condition $\hat{f}(0, v) = 1$. Following the procedure outlined in [13], the exact solution for $\hat{f}$ can be obtained. After expressing $\hat{f}(\tau, v)$ as

$$\hat{f}(\tau, v) = e^{h_1(\tau)v + h_2(\tau)}, \tag{3.2}$$

from Eq. (3.1), $h_1(\tau)$ and $h_2(\tau)$ are governed by

$$\begin{aligned} &h_1'(\tau) - (a_1 + a_4)h_1(\tau)^2 - a_2 h_1(\tau) - a_5 = 0 \ \text{ with } \ h_1(0) = 0, \\ &h_2'(\tau) - a_3 h_1(\tau) - a_6 = 0 \ \text{ with } \ h_2(0) = 0, \end{aligned}$$

and the solutions are

$$h_1(\tau) = \begin{cases} \frac{2a_5(e^{\sqrt{\Delta}\tau} - 1)}{a_2 + \sqrt{\Delta} - (a_2 - \sqrt{\Delta})e^{\sqrt{\Delta}\tau}}, & \text{for } \Delta > 0, \\ -\frac{2a_5\tau}{a_2\tau - 2}, & \text{for } \Delta = 0, \\ \frac{2a_5}{\sqrt{-\Delta}\left(\cot(\sqrt{-\Delta}\tau/2) - a_2/\sqrt{-\Delta}\right)}, & \text{for } \Delta < 0 \end{cases} \tag{3.3}$$

and

$$h_2(\tau) = \begin{cases} -\frac{a_3}{a_8}\log\left|\frac{(a_2 + \sqrt{\Delta})e^{-\sqrt{\Delta}\tau} - a_2 + \sqrt{\Delta}}{2\sqrt{\Delta}}\right| + \left(a_6 - \frac{a_3\sqrt{\Delta}}{2a_8} - \frac{a_2 a_3}{2a_8}\right)\tau, & \text{for } \Delta > 0, \\ -\frac{a_3}{a_8}\log\left|1 - \frac{a_2\tau}{2}\right| + \left(a_6 - \frac{a_2 a_3}{2a_8}\right)\tau, & \text{for } \Delta = 0, \\ -\frac{a_3}{a_8}\log\left|\cos\left(\frac{\sqrt{-\Delta}\tau}{2}\right) - \frac{a_2}{\sqrt{-\Delta}}\sin\left(\frac{\sqrt{-\Delta}\tau}{2}\right)\right| + \left(a_6 - \frac{a_2 a_3}{2a_8}\right)\tau, & \text{for } \Delta < 0, \end{cases} \tag{3.4}$$

where $a_8 = a_1 + a_4$ and $\Delta = a_2^2 - 4a_5 a_8$. From Eqs. (1.10) and (1.11), we obtain the exact solutions of optimal strategies $\hat{\phi^*}$ and $\hat{c^*}$ for the case of $\alpha = 0$:

$$\hat{\phi^*}_{exact} = \frac{A}{\gamma} + \rho\xi h_1(\tau), \tag{3.5}$$

$$\hat{c^*}_{exact}/w = 0, \tag{3.6}$$

where $h_1(\tau)$ is given by Eq. (3.3).

The exact solutions $\hat{f}_{exact}$, $\hat{\phi^*}_{exact}$ and $\hat{c^*}_{exact}$ for the case of $\alpha = 0$ given by Eqs. (3.2), (3.5) and (3.6) offer a benchmark for testing the accuracy of our numerical solutions. We show that our numerical solutions are accurate and efficient for $\alpha = 0$. Since neither the inhomogeneous term $\alpha^{\frac{1}{\gamma}}a_7$ nor the constant initial condition $(1-\alpha)^{\frac{1}{\gamma}}$ affects the stability or the accuracy of a numerical method, the accuracy and the stability of the method remain valid for $\alpha \neq 0$. The numerical results for $\alpha \neq 0$ are presented at the end of this section.

To set parameters for numerical validation, we use the estimation values of the parameters $\kappa, \theta, \xi, \rho, A, \beta$ and γ given in [4], [14], [5], [15] and the historical records of r. These values are listed in Table 1.

Table 1: Values for the parameters $\kappa, \theta, \xi, \rho, A, r, \beta, \gamma, a_c$ and a_w

parameter	κ	θ	ξ	ρ	A	r	β	γ	a_c	a_w
value	1.6048	0.0464	0.3796	-76.70%	1.55	1%	0.06	2	1	1

We note that, since a_w determines the wealth scale and a_w/a_c determines the temporal scale, without loss of generality, we choose $a_c = a_w = 1$ in this study.
For the range of the state variables t and v_t, we consider $T \leq 100$. Based on the historical records of Chicago Board Options Exchange Volatility Index, a popular measure of the implied volatility of S&P 500 index options, we examine the numerical solutions for the instantaneous volatility $\sigma_t = \sqrt{v_t}$ in the interval $[0.1, 0.8]$[1]. Since wealth w does not appear in Eqs. (2.1), (1.10) and (1.11), its value is irrelevant in our study.

In Table 2, we show the comparison between $\hat{f}_{exact}$, $\hat{f}_{num.}$ and $\hat{f}_{extpl.}$. $\hat{f}_{exact}$ is the exact solution of Eq. (3.1). When $\alpha = 0$, $\hat{f}_{num.}$ given by (2.8) is the numerical solution of Eq. (3.1) without performing Richardson's extrapolation and $\hat{f}_{extpl.}$ given by Eq. (2.15) is the numerical solution of Eq. (3.1) after performing Richardson's extrapolation. The relative errors in $\hat{f}_{num.}$ and $\hat{f}_{extpl.}$, namely, $|\frac{\hat{f}_{num.} - \hat{f}_{exact}}{\hat{f}_{exact}}|$ and $|\frac{\hat{f}_{extpl.} - \hat{f}_{exact}}{\hat{f}_{exact}}|$, are shown in the last two columns of Table 2.

Table 2: Comparison between the exact solution $\hat{f}_{exact}$, the numerical solution without Richardson's extrapolation $\hat{f}_{num.}$ and the numerical solution after Richardson's extrapolation $\hat{f}_{extpl.}$. The relative errors in numerical solutions are shown in the last two columns. Here the numerical solutions are obtained with $\Delta\tau = \Delta v = 0.01$.

τ	$\sigma = \sqrt{v}$	$\hat{f}_{exact}$	$\hat{f}_{num.}$	$\hat{f}_{extpl.}$	$\lvert\frac{\hat{f}_{num.} - \hat{f}_{exact}}{\hat{f}_{exact}}\rvert$	$\lvert\frac{\hat{f}_{extpl.} - \hat{f}_{exact}}{\hat{f}_{exact}}\rvert$
0.1	0.1	0.9961202112	0.9961202904	0.9961202112	7.9×10^{-8}	2.1×10^{-12}
0.1	0.4	0.9919378844	0.9919376809	0.9919378844	2.1×10^{-7}	6.2×10^{-12}
0.1	0.8	0.9786720788	0.9786706845	0.9786720789	1.4×10^{-6}	8.1×10^{-11}
1	0.1	0.9569365332	0.9569367514	0.9569365330	2.3×10^{-7}	1.3×10^{-10}
1	0.4	0.9339516557	0.9339511052	0.9339516557	5.9×10^{-7}	1.4×10^{-11}
1	0.8	0.8640451414	0.8640421623	0.8640451415	3.4×10^{-6}	8.9×10^{-11}
10	0.1	0.6062531863	0.6062531702	0.6062531839	2.6×10^{-8}	4.0×10^{-9}
10	0.4	0.5870696178	0.5870696136	0.5870696159	7.1×10^{-9}	3.2×10^{-9}
10	0.8	0.5296676303	0.5296676662	0.5296676288	6.8×10^{-8}	2.7×10^{-9}
100	0.1	0.0061763000	0.0061762986	0.0061762997	2.2×10^{-7}	4.2×10^{-8}
100	0.4	0.0059808642	0.0059808630	0.0059808640	2.0×10^{-7}	4.1×10^{-8}
100	0.8	0.0053960721	0.0053960714	0.0053960718	1.3×10^{-7}	4.0×10^{-8}

In Table 3, we show the comparison between the exact solution $\hat{\phi^*}_{exact}$ given by Eq. (3.5), the numerical solution $\hat{\phi^*}_{num.}$ determined from Eq. (2.11) without performing Richardson's extrapolation and $\hat{\phi^*}_{extpl.}$ determined from Eq. (2.16) after performing Richardson's extrapolation. The relative errors in $\hat{\phi^*}_{num.}$ and $\hat{\phi^*}_{extpl.}$, namely, $|\frac{\hat{\phi^*}_{num.} - \hat{\phi^*}_{exact}}{\hat{\phi^*}_{exact}}|$ and $|\frac{\hat{\phi^*}_{extpl.} - \hat{\phi^*}_{exact}}{\hat{\phi^*}_{exact}}|$, are shown in the last two columns of Table 3.

[1] To eliminate possible influence from the numerical boundary, we choose $v_{max} = 2$ in our numerical computations.

Table 3: Comparison between the exact solution $\hat{\phi}^*_{exact}$, the numerical solution without Richardson's extrapolation $\hat{\phi}^*_{num.}$ and the numerical solution after Richardson's extrapolation $\hat{\phi}^*_{extpl.}$. The relative errors in numerical solutions are shown in the last two columns. Here the numerical solutions are obtained with $\Delta\tau = \Delta v = 0.01$.

τ	$\sigma=\sqrt{v}$	$\hat{\phi}^*_{exact}$	$\hat{\phi}^*_{num.}$	$\hat{\phi}^*_{extpl.}$	$\left\vert\frac{\hat{\phi}^*_{num.}-\hat{\phi}^*_{exact}}{\hat{\phi}^*_{exact}}\right\vert$	$\left\vert\frac{\hat{\phi}^*_{extpl.}-\hat{\phi}^*_{exact}}{\hat{\phi}^*_{exact}}\right\vert$
0.1	0.1	0.7831667609	0.7831672706	0.7831667609	6.5×10^{-7}	1.5×10^{-11}
0.1	0.4	0.7831667609	0.7831673568	0.7831667609	7.6×10^{-7}	2.8×10^{-11}
0.1	0.8	0.7831667609	0.7831676487	0.7831667608	1.1×10^{-6}	9.1×10^{-11}
1	0.1	0.8221908761	0.8221924687	0.8221908741	1.9×10^{-6}	2.4×10^{-9}
1	0.4	0.8221908761	0.8221925174	0.8221908760	2.0×10^{-6}	6.6×10^{-11}
1	0.8	0.8221908761	0.8221927448	0.8221908760	2.3×10^{-6}	5.8×10^{-11}
10	0.1	0.8374121549	0.8374122233	0.8374121478	8.2×10^{-8}	8.6×10^{-9}
10	0.4	0.8374121549	0.8374121589	0.8374121544	4.8×10^{-9}	6.4×10^{-10}
10	0.8	0.8374121549	0.8374121565	0.8374121548	1.9×10^{-9}	1.6×10^{-10}
100	0.1	0.8374121969	0.8374122652	0.8374121897	8.2×10^{-8}	8.6×10^{-9}
100	0.4	0.8374121969	0.8374122009	0.8374121964	4.7×10^{-9}	6.4×10^{-10}
100	0.8	0.8374121969	0.8374121985	0.8374121968	1.9×10^{-9}	1.6×10^{-10}

There is no error in c^* since both numerical and theoretical values of c^* are zero.

In Table 4, we show the global relative errors and the computational times of $\hat{f}_{num.}$ and $\hat{f}_{extpl.}$ for different values of $\Delta\tau$ and Δv. The global relative error in $\hat{f}_{num.}$ is defined as the maximum of the local relative errors between $\hat{f}_{exact}$ and $\hat{f}_{num.}$ in the domain $0 \le \tau \le 100$ and $0 \le \sqrt{v} = \sigma \le 0.8$. The global relative error in $\hat{f}_{extpl.}$ is defined in the same way.

Table 4: Maximum relative errors and computational times of $\hat{f}_{num.}$ and $\hat{f}_{extpl.}$ in the domain $0 \le \tau \le 100$ and $0 \le \sqrt{v} = \sigma \le 0.8$.

step size		$\hat{f}_{num.}$		$\hat{f}_{extpl.}$	
$\Delta\tau$	Δv	max. rel. err.	comp. time	max. rel. err.	comp. time
0.02	0.02	1.5×10^{-5}	0.39	3.3×10^{-7}	0.84
0.01	0.01	3.8×10^{-6}	1.52	4.2×10^{-8}	3.31
0.005	0.005	9.6×10^{-7}	6.02	5.2×10^{-9}	13.18
0.0025	0.0025	2.4×10^{-7}	24.02	6.4×10^{-10}	52.54

Tables 4 confirms that our numerical solutions $\hat{f}_{num.}$ and $\hat{\phi}^*_{num.}$ have accuracy at orders of $\Delta\tau^2$, and that $\hat{f}_{extpl.}$ and $\hat{\phi}^*_{extpl.}$ have accuracy at orders of $\Delta\tau^3$. Therefore, the extrapolation technique does improve the accuracy. For the same order of accuracy, the application of Richardson's extrapolation technique significantly saves the computational time.

We have confirmed the accuracy of our numerical solutions for $\alpha = 0$. This guarantees that our numerical solution $f_{extpl.}$ for $\alpha \neq 0$ will also have an accuracy of $O(\Delta\tau^3)$. In Tables 5–7, we present the numerical solutions of $f_{extpl.}$, $\phi^*_{extpl.}$ and $c^*_{extpl.}/w$ for $\alpha = 0.1, 0.5$ and 0.9 with the step sizes $\Delta\tau = \Delta v = 0.001$. In Tables 8 and 9, we show the results for $\gamma = 1$ and 10 with $\alpha = 0.5$ and $\Delta\tau = \Delta v = 0.001$. All other parameter values in Tables 5–9 are the same as the ones in Table 1. All digits shown in Tables 5–9 are exact, in the sense that they do not change when we further refine the values of $\Delta\tau$ and Δv. Therefore, we have provided the first set of exact numerical data for optimal asset allocation and consumption strategies.

Table 5: Numerical solutions for $f, \phi^*, c^*/w$ after performing Richardson's extrapolation. Here $\alpha = 0.1$ and $\Delta\tau = \Delta v = 0.001$. Other parameter values are the same as those in Table 1.

τ	$\sigma=\sqrt{v}$	$f_{extpl.}$	${\phi^*}_{extpl.}$	${c^*}_{extpl.}/w$
0.1	0.1	0.976564446	0.783037744	0.323816587
0.1	0.4	0.972528955	0.783037394	0.325160258
0.1	0.8	0.959728271	0.783036268	0.329497187
1	0.1	1.217443571	0.817505616	0.259747370
1	0.4	1.191090583	0.817448428	0.265494305
1	0.8	1.110757357	0.817260409	0.284695630
10	0.1	3.073495770	0.833010306	0.102888629
10	0.4	2.983046823	0.832947671	0.106008315
10	0.8	2.711715965	0.832739549	0.116615372
100	0.1	6.227692985	0.835240145	0.050777674
100	0.4	6.037435062	0.835206694	0.052377833
100	0.8	5.467451073	0.835095101	0.057838243

Table 6: Numerical solutions for $f, \phi^*, c^*/w$ after performing Richardson's extrapolation. Here $\alpha = 0.5$ and $\Delta\tau = \Delta v = 0.001$. Other parameter values are the same as those in Table 1.

τ	$\sigma=\sqrt{v}$	$f_{extpl.}$	${\phi^*}_{extpl.}$	${c^*}_{extpl.}/w$
0.1	0.1	0.774937774	0.782803210	0.912469110
0.1	0.4	0.771828844	0.782802269	0.916144540
0.1	0.8	0.761966328	0.782799238	0.928002663
1	0.1	1.368974100	0.812872895	0.516523126
1	0.4	1.342553293	0.812781707	0.526688054
1	0.8	1.261787768	0.812484452	0.560400726
10	0.1	6.015181176	0.832382642	0.117553696
10	0.4	5.840061390	0.832312731	0.121078656
10	0.8	5.314527422	0.832080689	0.133051676
100	0.1	13.916810422	0.835238781	0.050809543
100	0.4	13.491657157	0.835205311	0.052410669
100	0.8	12.217961221	0.835093653	0.057874368

Table 7: Numerical solutions for $f, \phi^*, c^*/w$ after performing Richardson's extrapolation. Here $\alpha = 0.9$ and $\Delta\tau = \Delta v = 0.001$. Other parameter values are the same as those in Table 1.

τ	$\sigma=\sqrt{v}$	$f_{extpl.}$	${\phi^*}_{extpl.}$	${c^*}_{extpl.}/w$
0.1	0.1	0.409686387	0.782244153	2.315632955
0.1	0.4	0.408160455	0.782242030	2.324290085
0.1	0.8	0.403318534	0.782235205	2.352193658
1	0.1	1.231451909	0.808291630	0.770377869
1	0.4	1.210542696	0.808188721	0.783684294
1	0.8	1.146400499	0.807855696	0.827532175
10	0.1	7.686789813	0.832131657	0.123417359
10	0.4	7.463974966	0.832058953	0.127101618
10	0.8	6.795202083	0.831817743	0.139610756
100	0.1	18.667454349	0.835238326	0.050820175
100	0.4	18.097174990	0.835204850	0.052421624
100	0.8	16.388702420	0.835093170	0.057886419

Table 8: Numerical solutions for $f, \phi^*, c^*/w$ after performing Richardson's extrapolation. Here $\alpha = 0.5$, $\gamma = 1$ and $\Delta\tau = \Delta v = 0.001$. Other parameter values are the same as those in Table 1.

τ	σ=$\sqrt{v}$	$f_{extpl.}$	$\phi^*{}_{extpl.}$	$c^*{}_{extpl.}/w$
0.1	0.1	0.546859282	1.550000000	0.914311994
0.1	0.4	0.546859282	1.550000000	0.914311994
0.1	0.8	0.546859282	1.550000000	0.914311994
1	0.1	0.956177820	1.550000000	0.522915288
1	0.4	0.956177820	1.550000000	0.522915288
1	0.8	0.956177820	1.550000000	0.522915288
10	0.1	4.034308851	1.550000000	0.123936966
10	0.4	4.034308851	1.550000000	0.123936966
10	0.8	4.034308851	1.550000000	0.123936966
100	0.1	8.313916441	1.550000000	0.060140128
100	0.4	8.313916441	1.550000000	0.060140128
100	0.8	8.313916441	1.550000000	0.060140128

Table 9: Numerical solutions for $f, \phi^*, c^*/w$ after performing Richardson's extrapolation. Here $\alpha = 0.5$, $\gamma = 10$ and $\Delta\tau = \Delta v = 0.001$. Other parameter values are the same as those in Table 1.

τ	σ=$\sqrt{v}$	$f_{extpl.}$	$\phi^*{}_{extpl.}$	$c^*{}_{extpl.}/w$
0.1	0.1	1.024730849	0.157833805	0.910515178
0.1	0.4	1.023235912	0.157833681	0.911845432
0.1	0.8	1.018467192	0.157833282	0.916114922
1	0.1	1.840789858	0.169544364	0.506865565
1	0.4	1.827054636	0.169530453	0.510676021
1	0.8	1.783873146	0.169485620	0.523037747
10	0.1	9.182254704	0.178575988	0.101612624
10	0.4	9.071426914	0.178564461	0.102854049
10	0.8	8.726039146	0.178527017	0.106925144
100	0.1	38.546073475	0.180136006	0.024205656
100	0.4	38.050155049	0.180132872	0.024521135
100	0.8	36.506088964	0.180122662	0.025558284

Conclusion

We study the optimal asset allocation policy and the optimal consumption policy under CRRA utility function with stochastic volatility and finite investment horizons. So far, no numerical result is available in the literature for this important topic in portfolio management. In this paper, we present an accurate and efficient numerical method for solving the nonlinear partial differential equation which determines the values of optimal asset allocation and optimal consumption strategies. We use a three-level finite difference scheme, which has a second-order accuracy, to determine numerical solutions for these policies. We further improve our numerical solutions to a third-order accuracy by performing Richardson's extrapolation. Based on this method, we present the first set of accurate numerical solutions. Since optimal asset allocation and optimal consumption strategies under stochastic volatility are very important issues for portfolio management in modern finance, our method, presented in this paper for obtaining numerical results for these optimal strategies, will be very useful for further theoretical research and for applications in financial industry.

Appendix A. Derivations for $f_\tau(0, v)$, $f_{\tau\tau}(0, v)$ and $f_{\tau\tau\tau}(0, v)$

From Eq. (2.1), we obtain

$$f_\tau = a_1 v f_{vv} + (a_2 v + a_3) f_v + a_4 v \frac{f_v^2}{f} + (a_5 v + a_6) f + \alpha^{\frac{1}{\gamma}} a_7. \tag{A.1}$$

By taking the derivative of Eq. (A.1) with respect to τ, we obtain

$$f_{\tau\tau} = a_1 v f_{vv\tau} + (a_2 v + a_3) f_{v\tau} + a_4 v \frac{2 f_v f_{v\tau}}{f} - a_4 v \frac{f_v^2}{f^2} f_\tau + (a_5 v + a_6) f_\tau. \tag{A.2}$$

By taking the derivative of Eq. (A.2) with respect to τ, we obtain

$$\begin{aligned} f_{\tau\tau\tau} =& a_1 v f_{vv\tau\tau} + (a_2 v + a_3) f_{v\tau\tau} + a_4 v \frac{2 f_{v\tau}^2}{f} + a_4 v \frac{2 f_v f_{v\tau\tau}}{f} \\ & - 4 a_4 v \frac{f_v f_{v\tau}}{f^2} f_\tau + 2 a_4 v \frac{f_v^2}{f^3} f_\tau^2 - a_4 v \frac{f_v^2}{f^2} f_{\tau\tau} + (a_5 v + a_6) f_{\tau\tau}. \end{aligned} \tag{A.3}$$

By setting $\tau = 0$ in Eq. (A.1) and using the initial condition $f(0, v) = (1-\alpha)^{\frac{1}{\gamma}}$, we obtain

$$f_\tau(0, v) = (a_5 v + a_6) f(0, v) + \alpha^{\frac{1}{\gamma}} a_7. \tag{A.4}$$

By setting $\tau = 0$ in Eq. (A.2) and using the initial condition $f(0, v) = (1-\alpha)^{\frac{1}{\gamma}}$ and Eq. (A.4), we obtain

$$f_{\tau\tau}(0, v) = \left(a_5^2 v^2 + (a_2 a_5 + 2 a_5 a_6) v + a_3 a_5 + a_6^2\right) f(0, v) + (a_5 v + a_6) \alpha^{\frac{1}{\gamma}} a_7. \tag{A.5}$$

By setting $\tau = 0$ in Eq. (A.3) and using the initial condition $f(0, v) = (1-\alpha)^{\frac{1}{\gamma}}$, Eqs. (A.4) and (A.5), we obtain

$$\begin{aligned} f_{\tau\tau\tau}(0, v) =& \Big[a_5^3 v^3 + \left(3 a_2 a_5^2 + 3 a_6 a_5^2\right) v^2 + \left(a_5 a_2^2 + 3 a_5 a_6 a_2 + 2 a_1 a_5^2 + 3 a_3 a_5^2 + 2 a_4 a_5^2 + 3 a_5 a_6^2\right) v \\ & + a_2 a_3 a_5 + 3 a_3 a_6 a_5 + a_6^3\Big] f(0, v) + (a_2 v + a_3) a_5 \alpha^{\frac{1}{\gamma}} a_7 + (a_5 v + a_6)^2 \alpha^{\frac{1}{\gamma}} a_7. \end{aligned} \tag{A.6}$$

Acknowledgements

The work of Q. Zhang was supported by the Research Grants Council of the Hong Kong Special Administrative Region, China (project CityU 11335816).

REFERENCES

[1] A. Buraschi, P. Porchia, and F. Trojani, *Correlation risk and optimal portfolio choice*, The Journal of Finance, 65 (2010), pp. 393–420.

[2] H.-Y. Cao, Z.-Z. Sun, and G.-H. Gao, *A three-level linearized finite difference scheme for the camassa–holm equation*, Numerical Methods for Partial Differential Equations, 30 (2014), pp. 451–471.

[3] G. Chacko and L. M. Viceira, *Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets*, Review of Financial Studies, 18 (2005), pp. 1369–1402.

[4] P. Christoffersen, S. Heston, and K. Jacobs, *The shape and term structure of the index option smirk: Why multifactor stochastic volatility models work so well*, Management Science, 55 (2009), pp. 1914–1932.

[5] S. Frederick, G. Loewenstein, and T. O'donoghue, *Time discounting and time preference: A critical review*, Journal of economic literature, 40 (2002), pp. 351–401.

[6] K. R. French, G. W. Schwert, and R. F. Stambaugh, *Expected stock returns and volatility*, Journal of financial Economics, 19 (1987), pp. 3–29.

[7] D. He, *Exact solitary solution and a three-level linearly implicit conservative finite difference method for the generalized rosenau–kawahara-rlw equation with generalized novikov type perturbation*, Nonlinear Dynamics, (2016), pp. 1–20.

[8] D. He, *On the l-norm convergence of a three-level linearly implicit finite difference method for the extended fisher–kolmogorov equation in both 1d and 2d*, Computers & Mathematics with Applications, 71 (2016), pp. 2594–2607.

[9] S. L. Heston, *A closed-form solution for options with stochastic volatility with applications to bond and currency options*, Review of financial studies, 6 (1993), pp. 327–343.

[10] J. Kallsen and J. Muhle-Karbe, *Utility maximization in affine stochastic volatility models*, International Journal of Theoretical and Applied Finance, 13 (2010), pp. 459–477.

[11] T. S. Kim and E. Omberg, *Dynamic nonmyopic portfolio behavior*, Review of financial studies, 9 (1996), pp. 141–161.

[12] H. Kraft*, *Optimal portfolios and heston's stochastic volatility model: an explicit solution for power utility*, Quantitative Finance, 5 (2005), pp. 303–313.

[13] J. Liu, *Portfolio selection in stochastic environments*, Review of Financial Studies, 20 (2007), pp. 1–39.

[14] R. C. Merton, *On estimating the expected return on the market: An exploratory investigation*, Journal of financial economics, 8 (1980), pp. 323–361.

[15] C. Munk, *Financial asset pricing theory*, Oxford University Press, 2013.

[16] R. I. Palwasha, N. Ahmad, R. R. Ahmed, J. Vveinhardt, and D. Streimikien?, *Speed of mean reversion: an empirical analysis of kse, lse and ise indices*, Technological and Economic Development of Economy, 24 (2018), pp. 1435–1452.

[17] S. Raudys, A. Raudys, and i. Pabarskait?, *Sustainable economy inspired large-scale feed-forward portfolio construction*, Technological and Economic Development of Economy, 20 (2014), pp. 79–96.

[18] M. Schroder and C. Skiadas, *Optimal consumption and portfolio selection with stochastic differential utility*, Journal of Economic Theory, 89 (1999), pp. 68–126.

[19] J. A. Wachter, *Portfolio and consumption decisions under mean-reverting returns: An exact solution for complete markets*, Journal of financial and quantitative analysis, 37 (2002), pp. 63–91.

[20] B. Wongsaijai and K. Poochinapan, *A three-level average implicit finite difference scheme to solve equation obtained by coupling the rosenau–kdv equation and the rosenau–rlw equation*, Applied Mathematics and Computation, 245 (2014), pp. 289–304.

[21] Y. Xia, *Learning about predictability: The effects of parameter uncertainty on dynamic asset allocation*, The Journal of Finance, 56 (2001), pp. 205–246.